CNF Application in Discrete Optimization

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1 Introduction

A conjunctive normal form (CNF) can be interpreted as a search technology for the selecting of optimal solution in a discrete optimization problem (DOP) [1, 2]. By DOP we mean a following problem: to find the *optimal* solution $q^* \in Q$ such that $f(q^*) = \max\{f(q) \mid q \in Q\}$ for given finite solution domain Q and functional f.

In the well known branch-and-bound methods of implicit enumeration the domain Q is sequentially divided into non-intersecting subsets. Every subset is estimated in order to eliminate from further consideration such subsets that either optimal solution can not exist or a local optimal solution is found in them. We call them *shut* subsets.

The use of CNF allows us to *cover* the domain Q or some part of it by subsets. To do this, we associate with every clause of CNF a subset of Q and call the clause shut if its subset is shut. An *unsatisfiable* CNF with shut clauses covers the whole domain Q by shut subsets [1]. We can distinguish between the *primal* and the *dual* approaches. The primal approach involves consecutive construction of unsatisfiable CNF's that contain more and more shut clauses. The branch-and-bound method implements this approach by the use of a clause tree. Under the dual approach a CNF's with shut clauses are constructed step by step to be closer and closer to unsatisfiable CNF.

In this paper, firstly, we show that *pigeon-hole principle* gives ability to implement the primal approach with "non-treelike" unsatisfiable CNF's. Secondly, we suggest *complementary* formulas and a ways of their construction to embody the dual approach.

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In Section 2 we recall the description of DOP in suitable form and the interpretation of CNF in terms of DOP [1, 2]. Examples of non-treelike technology on the base of pigeon-hole principle are given in Section 3. Dual branch-and-bound method based on the complementary CNF is described in Section 4. Conclusion contains the comparison of branch-and-bound and dual branch-and-bound methods for two particular problems — KNAPSACK and PLANT LOCATION.

Further a CNF F is used in terms of *clause set*, where a clause $C \in F$ is a finite set of literals, and a literal $x \in C$ is a variable $x = p$ or a complemented variable $x = \bar{p}$. A clause is called *tautological* if it contains both a variable and its complemented variable.

2 DOP and interpretation of CNF by DOP

We will consider discrete optimization problem (DOP) in the following form.

Given n pairwise distinct finite sets B_i , $i \in 1{:}n$, and a functional f associating with every vector $q \in Q = B_1 \times B_2 \times \ldots \times B_n$ the value $f(q)$. The problem is how to find $q^* \in Q$ such that

$$f(q^*) = \max\{f(q) \mid q \in Q\}. \tag{1}$$

Any vector $q \in Q$ is called a *solution*, and q^* is called an *optimal solution*.

Many well known discrete problems can be presented as DOP. For example, in the KNAPSACK every set B_i consists of two alternatives: p_i — object i is selected, $\bar{p}_i$ — object i is not selected. If a solution $q \in Q$ satisfies the conditions of the knapsack problem then $f(q)$ is equal to the positive cost of all objects selected by solution q else $f(q) = 0$.

The PLANT LOCATION [3] can also be reduced to the form (1) where max is replaced by min. The role of sets B_i is played by the sets of possible construction schemes in each of the locations $i \in 1{:}n$. Value $f(q)$ defines the expenses involved according to the construction plan, i.e, the solution q.

Let I be a fixed subset of the set of indices $1{:}n$. Vector $\delta \in \prod_{i \in I} B_i$ is called a *partial solution* (hereinafter PS).

Since all sets $B_i, i \in 1{:}n$, are pairwise distinct then we can consider every solution–vector $q = q[1{:}n] \in B_1 \times B_2 \times \ldots \times B_n$ as the set $q = \{q[i] \mid i \in 1{:}n\}$ and every PS–vector $\delta = \delta[I] \in \prod_{i \in I} B_i$ as the set $\delta = \{\delta[i] \mid i \in I\}$.

A solution q is called an *extension* of a PS δ if $\delta \subseteq q$. With a PS $\delta = \{\delta[i] \mid i \in I\}$ we associate the set $L(\delta) = \{q \in Q \mid \delta \subseteq q\}$ of all its extensions.

Let D be a PS family and $L(D) = \cup_{\delta \in D} L(\delta)$. The family D is called *complete* if

$L(D) = Q$.

Suppose that every PS δ is evaluated by the estimation $E(\delta)$ so that it holds

$$E(\delta) \geq f(q)\ ,\ q \in L(\delta). \tag{2}$$

Let q_0 be a solution and $r = f(q_0)$. PS δ is called *shut* if

$$E(\delta) \leq r \tag{3}$$

and PS family D is called *shut* if every $\delta \in D$ is shut and $q_0 \in L(D)$. If D is shut then solution q_0 gives local optimum in the area $L(D)$, i.e.

$$f(q_0) = \max\{f(q) \mid q \in L(D)\}.$$

If we were fortunate to construct PS family D which turns out both shut and complete then DOP would be solved and q_0 would be an optimal solution.

It follows that we can use two approaches for the search of an optimal solution: *primal* and *dual*. In primal (resp. dual) approach complete (resp. shut) PS families D are constructed step by step and process come to a stop when current D becomes shut (resp. complete).

Well known branch-and-bound method realizes primal approach in which each family D is the family of all branches in PS tree. We will use now a DOP interpretation of CNF to construct "non-treelike" PS families in primal approach and to realize dual approach.

We shall restrict our consideration to the case DOP when $B_i = \{p_i,\ \bar{p}_i\}\ ,\ i \in 1{:}\,n$. Then we can interpret a clause set F over n variables $p_i\ ,\ i \in 1{:}\,n$, without tautological clauses as a PS family F. It easy to see that

PS family F is complete $\iff$ clause set F is unsatisfiable.

Indeed, consider solution $q \in Q$ as the set of literals with *false* value. Then completeness of family F is the same that every valuation is a refutation for F.

Now suppose that we have a concrete DOP with sets $B_i = \{p_i, \bar{p}_i\}\ ,\ i \in 1{:}\,n$, (for example, KNAPSACK). Then any unsatisfiable formula F generates the complete PS family F. Moreover, let there exists a polynomial method for calculation of PS estimation in accord with (2). If there is solution $q_0 \in L(F)$ with value $r = f(q_0)$ such that inequality (3) holds for every $\delta \in F$ then formula F is the proof that q_0 is optimal solution.

The set of all branches in a PS tree of the branch-and-bound method gives an example of the "treelike" unsatisfiable clause set. And if a branch is non-shut then this branch is to be extended, at best, by two ways in an attempt to get a closer estimation. It would be good to have a "non-treelike" complete PS families with a small number of long PS in the hope that every PS can be estimated to be shut. Every "hard" formula affords such an opportunity. Moreover, we know how to construct the initial data for KNAPSACK problem that will be solved by a non-treelike technology much faster than by a branch-and-bound.

3 Examples of non-treelike complete families

At first we will consider the propositional version of *pigeon-hole principle* which can be formulated in terms of the graph coloring by the following way.
Let $G = \langle V, E \rangle$ be a graph and k be a fixed positive integer. Associate with graph G and number k the clause set $PH_{G,k} = PH^+_{G,k} \bigcup PH^-_{G,k}$ where

$$PH^+_{G,k} = \{\{v_{ij} \mid j \in 1:k\} \mid i \in V\}$$

$$PH^-_{G,k} = \{\{\bar{v}_{i_1 j}, \bar{v}_{i_2 j}\} \mid (i_1, i_2) \in E,\ j \in 1:k\}.$$

Variables v_{ij} has the meaning "vertex i is coloring by color j". Positive part states that every vertex is colored, and negative part states that two vertices of one arc can not be coloring by the same color. Thereby, we have.

Proposition 1 *Graph G is colorable by k colors $\iff$ $PH_{G,k}$ is satisfiable.*

It is obviously that a complete graph G with n vertices is uncolorable by $n-1$ colors and thus $PH_{G,n-1}$ is unsatisfiable. The following theorem is known for this case.

Theorem 1 (Haken [4]). *There is $c > 1$ so that any resolution refutation of $PH_{G,n-1}$ contains at least c^{n-1} distinct clauses.*

Thereby, every PS tree of branch-and-bound method such, that every its branch contains some clause from $PH_{G,n-1}$, has at least c^{n-1} branches. At the same time, the number of sets in $PH_{G,n-1}$ is about $O(n^3)$. In the given sense a complete graph G generates a "non-treelike" complete PS family $PH_{G,n-1}$. And every graph G which is uncolorable by k colors generates a complete PS family $PH_{G,k}$. But there are many two-member PS in such families.

We present now the following **technique for the clause lengthening** in $PH_{G,k}$ without increase of their number.

1. Associate with i-clause $\{v_{ij} \mid j \in 1:k\}$ from the positive part of $PH_{G,k}$ the minimal unsatisfiable saturated Horn formula with clause set

$$H_i = \{\{v_{ij} \mid j \in 1:k\}; \{v_{ij} \mid j \in 1:k-1\} \cup \{\bar{v}_{ik}\}; \ldots; \{\bar{v}_{i1}\}\}$$

2. Replace every literal in the negative part of $PH_{G,k}$ by its corresponding clause from H_i so to obtain the clause set

$$PHH^{-}_{G,k} = \{\{C(\bar{v}_{i_1j}) \cup C(\bar{v}_{i_2j}) \mid (i_1, i_2) \in E,\ j \in 1:k\}$$

where $C(\bar{v}_{ij}) = \{v_{it} \mid t \in 1:j-1\} \cup \{\bar{v}_{ij}\}$.

Proposition 2 *Clause set $PH^{+}_{G,k} \bigcup PHH^{-}_{G,k}$ is unsatisfiable.*

Definition 1 *The family $\Omega = \{D_1, \ldots, D_m\}$ of clause sets we call a* ***plait****. Every $\omega \in D_1 \times \ldots \times D_m$ is a sequence $\omega = \{C_1, \ldots, C_m\}$ of clauses. Clause set*

$$P(\Omega) = \{C(\omega) \mid \omega \in D_1 \times \ldots \times D_m\} \textit{ where } C(\omega) = \bigcup_{i=1,\ldots,m} C_i$$

is called the ***product*** *of plait Ω.*

The **constructing of complete PS families** on the base of pigeon-hole principle with the using of product can be done by following way. Let be given graph $G = \langle V, E \rangle$.

1. Associate with every vertex $i \in V$ an unsatisfiable clause set F_i.
For example, F_i can be an unsatisfiable Horn formula or a minimal unsatisfiable basic (with n variables and $n+1$ clauses) formula [5].

2. Divide every clause set F_i into k clause subsets $F_{ij}, j \in 1:k$, and construct new clause set

$$PHV_{G,k} = \bigcup_{(i_1,i_2)\in E,\ j\in 1:k} P_{(i_1,i_2)j}$$

where $P_{(i_1,i_2)j}$ is the product of plait $\{F_{i_1j}, F_{i_2j}\}$.

Proposition 3 *$PHV_{G,k}$ is unsatisfiable if graph G is uncolorable by k colors.*

In terms of DOP a plait is a family of PS families. Call every PS family in a plait as PS *bunch*. Suppose that we have a DOP in which it is possible to estimate every PS bunch D by $E(D)$ and every plait Ω by $E(\Omega)$ so that

$$E(D) \geq f(q)\ ,\ q \in L(D) = \bigcup_{\delta \in D} L(\delta)$$

$$E(\Omega) \geq f(q)\ ,\ q \in L(\Omega) = \bigcup_{\delta \in P(\Omega)} L(\delta)$$

where $P(\Omega)$ is the product of plait Ω. Then inequalities

$$E(D) \leq r;\ E(\Omega) \leq r$$

define respectively that bunch D or plait Ω is shut by the same way as inequality (3) defines that PS is shut. On this base we can construct **plait-and-bound method** in which the role of branch is played by plait.

Non-treelike technology can be used here also. For example, let we have DOP with sets $B_t = \{p_t, \bar{p}_t\}$, $t \in 1\colon n$. Divide the indices set $1\colon n$ into l parts N_i , $i \in 1\colon l$. Consider an uncolorable by k colors graph $G = \langle V, E \rangle$ with $|V| = l$ and apply the **constructing of complete PS families** where the associating with vertex i unsatisfiable clause set F_i is defined on variables p_t , $t \in N_i$. Now we can estimate both every PS bunch $F_{i_1 j}$, $F_{i_2 j}$ and plait $\{F_{i_1 j}, F_{i_2 j}\}$ before to construct the product of this plait.

The plait technique is appropriate for problems in which partial solutions can be joined in a bunch with a common characteristic by semantic way. For example, well known *quadratic assignment problem* is a such problem. We can define a bunch here by introducing bans on some assignments.

4 Dual branch-and-bound method

Suppose that a shut PS family D can be constructed by some analysis of a concrete DOP. In order to verify that D is complete we will use a *measure*

$$\mu(D) = \frac{|L(D)|}{2^n}$$

where $L(D) = \cup_{\delta \in D} L(\delta)$ and $L(\delta)$ is the set of all extensions of PS δ. If we consider D as the clause set then $\mu(D)$ is the measure of refuting valuations for D. If $\mu(D) = 1$ then formula D is unsatisfiable and hence the PS family D is complete. The ways of the measure calculation are developed, for example, in [6, 7]. But our main interest here is how to add new PS to family D for the purpose of the obtaining of a complete family.

Definition 2 *PS family $C(D)$ is called* ***complementary*** *to PS family D if*

$$\mu(D \cup C(D)) = 1 \tag{4}$$

and

$$\mu(D \cup C(D)) = \mu(D) + \mu(C(D)). \tag{5}$$

Equality (5) means that every PS from $C(D)$ is contradictory with every PS from D and hence $L(D) \cap L(C(D)) = \emptyset$.

Definition 3 *Let D be a clause set. $R(D)$ is called **refined** D if $R(D)$ is obtained from D by deleting both of all tautological clauses and of all such clauses which are containing a clause from D.*

It is obviously that $\mu(D) = \mu((R(D))$. Present now two ways of the construction of a complementary family.

1. With a PS δ from given family D associate the family $I(\delta) = \{I(a) \mid a \in \delta\}$ where $I(a)$ is the complementary literal to literal a in respective set B_i (if $a = p_i$ then $I(a) = \bar{p}_i$ and if $a = \bar{p}_i$ then $I(a) = p_i$). Let $I(D)$ be the refined product of plait $\{I(\delta) \mid \delta \in D\}$.

Proposition 4 *$I(D)$ is a PS family and it is complementary to D.*

2. With a PS δ from given family D associate the clause set $H(\delta)$ of the minimal unsatisfiable saturated (maybe renamed) Horn formula in which δ is one of two longest clauses. Let $H(D)$ be the refined product of plait $\{(H(\delta) \setminus \{\delta\}) \mid \delta \in D\}$.

Proposition 5 *$H(D)$ is a PS family and it is complementary to D.*

Please note that $\mu(H(D)) = \sum_{\delta \in H(D)} \mu(\delta)$ because all PS in $H(D)$ are pairwise contradictory, whereas such is not the case for $I(D)$ (in general). **Dual branch-and-bound** method can be described as follows. Let D be a shut PS family. Construct by some way a complementary family $C = C(D)$.

1. Estimate every PS from C. Let S be the family of all shut PS in C and NS be the family of all non-shut PS in C.

2. If NS is empty then *stop*.

3. If S is empty then some PS $\delta \in NS$ is lengthened to shut PS β. Design $\delta' = \beta \backslash \delta$. Let R_1 be the refined product of plait $\{\{\delta\}, I(\delta')\}$ and R_2 be the refined product of plait $\{NS \setminus \{\delta\}, I(\beta)\}$. New iteration is beginning with $C = R_1 \cup R_2$.

4. If both S and NS are non-empty then construct refined product R of plait $\{NS, C(S)\}$ where $C(S)$ is a complementary to S family. New iteration is beginning with $C = R$.

Proposition 6 *Both $R_1 \cup R_2$ from 3 and R from 4 are PS families;*
$R_1 \cup R_2$ is complementary to $D \cup \{\beta\}$; R is complementary to $D \cup S$.

5 Conclusion

Some realizations of a branch-and-bound and a dual branch-and-bound methods for KNAPSACK and PLANT LOCATION were compared in [8]. The results of this experiments show that for KNAPSACK both methods work approximately equally for every generated problem with 50–1000 items. But for PLANT LOCATION the dual branch-and-bound is faster than the other method for every generated problem with 10–100 locations and 10–30 consumers. This is because the best branching index in each node of the PS tree can be defined for KNAPSACK, but for PLANT LOCATION there is the Erlencotter estimation [3] giving a *set* of best indices as the ground for the dual method. The task to determine which of the two approaches is appropriate for a given discrete optimizatiom problem opens a great field for further development.

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