Exact Selection of Minimal Unsatisfiable Subformulae for Special Classes of Propositional Formulae

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Extended Abstract

A *minimal unsatisfiable subformula* (MUS) of a given propositional formula is a subset of clauses which is unsatisfiable, but becomes satisfiable as soon as we remove any of its clauses. In several applicative fields, a relevant problem besides solving the satisfiability problem, is to detect a minimal unsatisfiable subformula. This is expecially true when the application is encoded into a propositional formula which should have a well-defined satisfiability property (either to be satisfiable or to be unsatisfiable). While selection of a MUS is a hard problem in general, we show classes of formulae for which such individuation can be performed efficiently.

Given an $m \times n$ rational matrix A and a rational m-vector b, consider the (primal) system of linear inequalities

$$\{Ax \leq b \tag{1}$$

and the (dual) system of linear inequalities

$$\begin{cases} y^T A = 0 \\ y^T b < 0 \\ y \geq 0 \end{cases} \tag{2}$$

A variant of the well-known Farkas' lemma (see for instance [2]) for systems of linear inequalities asserts that exactly one of the following holds: (1) is feasible, (2) is feasible.

Basing on this, Gleeson and Ryan [4] showed that an *irreducible infeasible subsystem* (IIS, see e.g. [1]) of a given system of linear inequalities can be selected by solving the new (dual) system

$$\begin{cases} y^T A = 0 \\ y^T b \leq -1 \\ y \geq 0 \end{cases} \tag{3}$$

If (3) is infeasible, (1) is feasible. On the contrary, if (3) is feasible, (1) is infeasible and each IIS is given by the positive components of each vertex of the polyhedron. Therefore, checking the feasibility of (1), and, if infeasible, identifying one of its IIS, becomes the problem of finding a vertex of a polyhedron. Unfortunately, the above can be not true in the case we want an integer solutions for (1). In such case, if (3) is feasible, (1) is infeasible, but if (3) is infeasible, (1) can still not have any integer solution. A result analogous to the continuous case would hold, however, when there is a guarantee that the polyhedron (1), if nonempty, contains at least one integer point.

Coming back to satisfiability, it is well-known that each clause $(\alpha_i \vee ... \vee \alpha_j \vee \neg\alpha_k \vee ... \vee \neg\alpha_n)$ can be expressed as a linear inequality by defining the incidence vector of the set of positive propositions a^π and that one of the negative propositions a^ν, as follows

$$\sum_{i=1}^{n} a_i^\pi x_i + \sum_{i=1}^{n} a_i^\nu (1 - x_i) \geq 1$$

A system of linear inequalities with binary variables is therefore obtained. There are several classes of formulae for which such system has a linear relaxation defining a polyhedron that, if nonempty, contains at least one integer point. The structure of arising polytopes is studied also in [3]. Examples are Horn formulae, Balanced formulae, etc. Most of them are already known to be easy cases for the satisfiability problem (see [3, 5, 7] for extensive references). Some results based on the construction of the dual linear program are already presented in [6].

A system of linear inequalities which plays the function of (3) can therefore be obtained. Hence, the problem of testing satisfiability and, if unsatisfiable, identifying a MUS, can be solved by finding a vertex of a polyhedron. Validation of a Horn knowledge base could be performed this way.

Keywords: Consistency restoring, MUS selection.

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