Minimal Falsity for QBF with Deficiency One

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Abstract

We consider the minimal falsity problem (*MF*) for *QCNF*. A *QCNF*
formula is called minimal false, if the formula is false and any proper subformula is true. The deficiency of a *QBF* formula is defined as the difference between the number of clauses and the number of existential variables. In this paper we show that the minimal falsity problem for *QCNF* formulas with deficiency 1 is solvable in polynomial time.

1 Introduction

Quantified Boolean formulas gain their importance since they are suitable for the representation of many problems i.e. planning, abduction, and non-monotonic reasoning. We are interested in closed quantified Boolean formulas with matrix in conjunctive normal form (*QCNF*). A *QCNF* formula is said to be minimal false if the formula is false and any proper subformula is true. This notion is a generalization of minimal unsatisfiable *CNF* formulas. A *CNF* formula is minimal unsatisfiable if the formula is unsatisfiable and any proper subformula is satisfiable. Although the minimal unsatisfiability problem is D^P–complete [7], the problem can be solved in polynomial time for formulas with fixed deficiciency [6]. The deficiency of a *CNF* formula is the difference between the number of clauses and the number of variables. It is an open question whether the minimal falsity problem for *QCNF* formulas with fixed deficiency is solvable in polynomial time, where the deficiency of a *QCNF* formula is defined as the difference between the number of clauses and the number of existential variables. In [3] some very restricted subclasses are proved to be solvable in polynomial time. The main task of this paper is to prove that the minimal falsity problem for *QCNF* formulas with deficiency 1 is tractable.

2 Notations and Preliminaries

A propositional formula in conjunctive normal form (*CNF*) is a conjunction of clauses. *CNF* formulas will be considered as multi–sets of clauses. Thus, they may contain multiple occurrences of clauses. Clauses are disjunctions of literals. We demand that any clauses contains no pair of complementary literals. A literal is a variable or a negated variable. If a *CNF* formula F over n variables has $n+k$ clauses, then we say k is the *deficiency* of F. Often we write $d(F)$ for the deficiency.

A *CNF* formula is termed *minimal unsatisfiable*, if the formula is unsatisfiable and removing an arbitrary clause leads to a satisfiable formula. The set of minimal unsatisfiable formulas is denoted as *MU*. For fixed k the class *MU(k)* is the set of *MU* formulas with deficiency k.

Given a *CNF* formula F and a variable x occurring in F. $F[x = 0]$ resp. $F[x = 1]$, denotes the formula obtained from F by deleting the clauses containing $\neg x$ resp. x and removing all occurrences of x resp. $\neg x$.

QBF denotes the set of quantified Boolean formulas, where the formula is given in prefix normal form and the matrix is a propositional formula. For sake of simplicity we only consider closed quantified Boolean formulas, that is, formulas do not contain free variables. That means any variable is bounded by a quantifier. *QCNF* is the set of formulas for which the matrix is a *CNF* formula. Any *QCNF* formula Φ has the form $\Phi = Q_1x_1 \cdots Q_nx_n\phi$, where $Q \in \{\exists, \forall\}$ and ϕ is a *CNF* formula. Sometimes we use an abbreviation and write $\Phi = Q\phi$.

An occurrence of a literal x or $\neg x$ is called universal resp. existential, if this specific occurrence of x is bounded by an universal quantifier resp. by an existential quantifier. Sometimes universally resp. existentially quantified variables are denoted as $\forall$– resp. $\exists$–variables.

A quantified boolean formula $\Phi \in$ *QCNF* is termed *minimal false*, if Φ is false and after removing an arbitrary clause the resulting formula is true. The set of minimal false formulas is denoted as *MF*. As an analog to minimal unsatisfiable formulas with fixed deficiency *MU(k)* we will introduce the classes *MF(k)*. For calculating the deficiency of a *QCNF* formula we do not count all the variables, but only the existential variables.

Suppose we have a formula $Q\phi \in$ *QCNF* with $\exists$–variables $x_1, \cdots, x_n$. Then $\phi_{|\exists}$ is the conjunction of clauses we obtain after removing all occurrences of universal literals in ϕ. Please note, that the propositional formula $\phi_{|\exists}$ may contain multiple occurrence of clauses. If a clause in ϕ contains only universal literals then the result is the empty clause $\sqcup$. Clearly, any *QCNF* formula containing an universal clause is false. Hence we only consider *QCNF* formulas without universal clauses.

We define for fixed k the class *MF(k)* to be the set of *MF* formulas $\Phi = Q\phi$, where the deficiency of the matrix $\phi_{|\exists}$ is k. For the deficiency we write $d(\Phi) = k$. We define $d^*(\phi) = \max\{d(\phi') \mid \phi' \subseteq \phi\}$.

Let $\Phi = Q_1x_1 \cdots Q_nx_n\phi$, $\Phi' = Q_1x_1 \cdots Q_nx_n\phi'$ be two *QCNF* formulas. We say Φ' is a subformula of Φ, denoted as $\Phi' \subseteq \Phi$, if ϕ' is a subformula of ϕ. Here $\subseteq$ stands for the inclusion of multi-sets.

3 QCNF Formulas in MF(1)

We first recall some known results on minimal false formulas.

Theorem 1 *[3]*
1. If $\Phi \in MF$ then the deficiency $d(\Phi)$ is greater than 0.
2. There is a formula $\Phi = Q\phi \in MF$ for which $\phi_{|\exists}$ is not minimal unsatisfiable.
3. $\Phi \in MF(1) \Longrightarrow \phi_{|\exists} \in MU(1)$.

The theorem shows that for minimal false *QCNF* formulas $\Phi = Q\phi$ with deficiency 1 the existential part $\phi_{|\exists}$ must be in *MU(1)* . Since *MU(1)* can be decided in polynomial time and any proper subformula of a *MU(1)* formula is satisfiable, the main step in our procedure is the test whether the formula Φ is false. For this test we make use of the well-known structure of *MU(1)* formulas.

Lemma 1 *([2]) For any formula $\phi \in MU(1)$ there is a variable $x \in var(\phi)$ such that ϕ can be partitioned into two disjoint subformula ϕ_1, ϕ_2:*
(1) ϕ_1 resp. ϕ_2 contains no occurrence of $\neg x$ resp. x,
(2) $\phi_1[x = 0] \in MU(1)$, $\phi_2[x = 1] \in MU(1)$,
(3) $\phi_1[x = 0]$, $\phi_2[x = 1]$ have distinct variables, that is, $var(\phi_1[x = 0]) \cap var(\phi_2[x = 1]) = \emptyset$.

Let ϕ, x, ϕ_1, ϕ_2 be as in Lemma 1, then we call $(\phi_1[x = 0], \phi_2[x = 1])$ a disjunctive splitting of ϕ on x. Usually, we use $(\phi^x, \phi^{\neg x})$ for a splitting of ϕ on x.

Suppose $\Phi = Q\phi$ is a *QCNF* formula such that $\phi_{|\exists} \in MU(1)$. From a disjunctive splitting of $\phi_{|\exists}$ we recover the occurrences of all universal literals, the resulting splitting is also called a disjunctive splitting of ϕ. We also use $(\phi^x, \phi^{\neg x})$ to denote a splitting of ϕ on x.

Suppose $\Phi = Q\phi$ is a *QCNF* formula such that $\phi_{|\exists} \in MU(1)$. Then we can construct a (disjunctive) splitting tree, such that

(1) the root is the formula ϕ.

(2) For any inner node ψ, its two children form a disjunctive splitting of ψ on a variable x, and the two edges are labeled by x and $\neg x$, respectively.

(3) Each leaf contains only one universal clause (a clause containing no existential literals is called a universal clause).

Please note that for any $\Phi = Q\phi$ with $\phi_{|\exists} \in MU(1)$ we can find in polynomial time a variable x such that the splitting of ϕ on x is disjunctive. Consequently, we can construct a splitting tree of ϕ in polynomial time.

Example 1. Let $\Phi = \exists x_1 \exists x_3 \forall y \exists x_2 \exists x_4\ \phi$, where ϕ is the following formula. The columns of the matrices are the clauses of the formula:

$$\phi = \left\{ \begin{array}{ccccc} x_1 & \neg x_1 & \neg x_3 & x_3 & x_4 \\ \neg x_2 & y & x_2 & \neg y & \\ & \neg x_2 & & \neg x_4 & \end{array} \right\}$$

$$\phi_{|\exists} = \left\{ \begin{array}{ccccc} x_1 & \neg x_1 & \neg x_3 & x_3 & x_4 \\ \neg x_2 & & x_2 & & \\ & \neg x_2 & & \neg x_4 & \end{array} \right\}$$

Now we split on the existential variable x_2 and obtain

$$\phi^{x_2}_{|\exists} = \left\{ \begin{array}{ccc} \neg x_3 & x_3 & x_4 \\ & \neg x_4 & \end{array} \right\}, \qquad \phi^{\neg x_2}_{|\exists} = \left\{ \begin{array}{cc} x_1 & \neg x_1 \end{array} \right\}$$

Adding the universal variables we get

$$\phi^{x_2} = \left\{ \begin{array}{ccc} \neg x_3 & x_3 & x_4 \\ & \neg y & \\ & \neg x_4 & \end{array} \right\}, \qquad \phi^{\neg x_2} = \left\{ \begin{array}{cc} x_1 & \neg x_1 \\ & y \end{array} \right\}$$

Let $\Phi = Q\phi$ be a *QCNF* formula. Suppose x is a $\exists$–variable and y is a $\forall$–variable. If in the prefix $\forall y$ is on the left of $\exists x$ then we write $y < x$.

For the proof of our main theorem and the algorithm we need two Lemmata, which will be shown in the next section.

Lemma 2 *Let $\Phi = Q\phi$ be a QCNF formula such that $\phi_{|\exists} \in MU(1)$. Suppose $C_1 = (\cdots y \vee L \vee \cdots)$, $C_2 = (\cdots \neg y \vee \neg L \vee \cdots)$ are two clauses in ϕ, where $L \in \{x, \neg x\}$. If y is a $\forall$–variable, x is a $\exists$–variable, and $y < x$, then Φ is true.*

Example 2: Let $\Phi = \exists x_1 \exists x_2 \forall y \exists x_3 \ \phi$, where ϕ is the following formula.

$$\phi = \left\{ \begin{array}{cccc} x_1 & \neg x_1 & x_2 & \neg x_2 \\ y & & & \neg y \\ x_3 & x_3 & \neg x_3 & \neg x_3 \end{array} \right\}$$

For $x \leftrightarrow \neg y$ the first and the last clause are always true. Therefore, for $x_1 = 0$ and $x_2 = 1$ the formula Φ is true.

Lemma 3 *Let $\Phi = Q\phi$ be a QCNF formula such that $\phi_{|\exists} \in MU(1)$. Suppose the case in Lemma 2 does not occur. Then Φ is true if and only if in any splitting tree of ϕ there is a formula ψ such that there is a $\forall$–variable y:*

(1) For all $x \in var(\psi_{|\exists})$, $y < x$.

(2) There are clauses f_1 and f_2 in the two children of ψ, respectively, such that $y \in f_1$ and $\neg y \in f_2$.

Example 3. Let $\Phi = \exists x_1 \forall y_1 \forall y_2 \exists x_2 \exists x_3 \exists x_4 \exists x_5\ \phi$, where ϕ is the following formula

$$\phi = \left\{ \begin{array}{cccccc} x_2 & y_1 & \neg y_1 & \neg x_3 & y_2 & \neg x_1 \\ x_4 & x_1 & x_3 & \neg x_4 & \neg x_1 & \neg x_5 \\ & \neg x_2 & & & x_5 & \end{array} \right\}$$

It is easy to see that $\phi_{|\exists} \in$ *MU(1)* . Now splitting ϕ on x_1 we get

$$\phi^{x_1} = \left\{ \begin{array}{cccc} x_2 & y_1 & \neg y_1 & \neg x_3 \\ x_4 & \neg x_2 & x_3 & \neg x_4 \end{array} \right\}, \quad \phi^{\neg x_1} = \left\{ \begin{array}{cc} y_2 & \\ x_5 & \neg x_5 \end{array} \right\}$$

Then split ϕ^{x_1} on x_4 , we get

$$(\phi^{x_1})^{x_4} = \left\{ \begin{array}{cc} x_2 & y_1 \\ & \neg x_2 \end{array} \right\}, \quad (\phi^{x_1})^{\neg x_4} = \left\{ \begin{array}{cc} \neg y_1 & \neg x_3 \\ x_3 & \end{array} \right\}$$

Please note that y occurs in a clause of $(\phi^z)^x$ and $\neg y$ occurs in a clause of $(\phi^z)^{\neg x}$. Since y is a $\forall$–variable and $y < x_2$, $y < x_3$ and $y < x_4$, it follows from Lemma 3 that Φ is true.

Example 4. Let $\Phi = \exists x_1 \exists x_3 \forall y \exists x_2 \exists x_4\ \phi$, where ϕ is the following formula.

$$\phi = \left\{ \begin{array}{ccccc} x_1 & \neg x_1 & \neg x_3 & x_3 & x_4 \\ \neg x_2 & y & x_2 & \neg y & \\ & \neg x_2 & & \neg x_4 & \end{array} \right\}$$

$$\phi_{|\exists} = \left\{ \begin{array}{ccccc} x_1 & \neg x_1 & \neg x_3 & x_3 & x_4 \\ \neg x_2 & & x_2 & & \\ & \neg x_2 & & \neg x_4 & \end{array} \right\}$$

Clearly, $\phi_{|\exists} \in$ *MU(1)* and the case of Lemma 2 does not occur. Now splitting ϕ on x_2 we obtain

$$\phi^{x_2} = \left\{ \begin{array}{ccc} \neg x_3 & x_3 & x_4 \\ & \neg y & \\ & \neg x_4 & \end{array} \right\}, \qquad \phi^{\neg x_2} = \left\{ \begin{array}{cc} x_1 & \neg x_1 \\ & y \end{array} \right\}$$

Although y and $\neg y$ occur in the two splitting formulas respectively, $\forall y$ is on the right to $\exists x_1$ and $\exists x_3$. Thus, this is not the case of Lemma 3. Please note that no splitting formula contains complementary universal literals. Hence further splitting can not lead to the case of Lemma 3. Consequently, Φ is false.

We shall prove the above two Lemmata in the next section. Based on Lemma 2 and 3 we establish an algorithm which decides whether a *QCNF* formula is in *MF(1)* .

Algorithm $mf(1)$

Input: A *QCNF* formula Φ

Output: YES, if $\Phi \in$ *MF(1)* and NO, otherwise

begin

if $d(\Phi) \neq 1$ **then** return NO

if $\Phi_{|\exists}$ is not in *MU(1)* **then** return NO

if the case of Lemma 2 occurs **then** return NO

compute a disjunctive splitting tree of ϕ

if the case of Lemma 3 occurs **then** return NO

return YES

end

Please note that any *QCNF* formula is true whenever its existential part is satisfiable. Now given a formula $Q\phi$ such that $\phi_{|\exists} \in$ *MU(1)* , then $Q\phi$ is minimal false if and only if it is false. Therefore, to determine whether a formula $Q\phi$ is in MF(1), first by Theorem 1 check if $\phi_{|\exists} \in$ *MU(1)* . If this condition holds then check whether the case of Lemma 2 occurs. If so then the formula is true and return no. Otherwise, compute one disjunctive splitting tree for ϕ. If the conditions (1)–(2) of Lemma 3 hold then by Lemma 3 $Q\phi$ is true and return no; otherwise, $Q\phi$ is false and return yes. Please notice that to test if $Q\phi$ is false it is sufficient to compute one disjunctive splitting tree. Because the test for *MU(1)* , the computation of a disjunctive splitting tree, and the applications of Lemma 2 and 3 can be performed in polynomial time, the algorithm $mf(1)$ requires not more than polynomial time. Thus, we obtain the following theorem

Theorem 2 *The minimal falsity problem for QCNF formulas with deficiency 1 is solvable in polynomial time.*

4 Proofs of Lemma 2 and 3

Lemma 2 *Let* $\Phi = Q\phi$ *be a QCNF formula such that* $\phi_{|\exists} \in$ *MU(1) . Suppose* $C_1 = (\cdots y \vee L \vee \cdots)$, $C_2 = (\cdots \neg y \vee \neg L \vee \cdots)$ *are two clauses in* ϕ, *where* $L \in \{x, \neg x\}$. *If* y *is a* $\forall$*–variable,* x *is a* $\exists$*–variable, and* $y < x$, *then* Φ *is true.*

Proof: We first remove any $\forall$–variable different from y. If the resulting formula is true then the initial formula Φ is true. In a next step we move any $\exists$–quantifier different from $\exists x$ in front of $\forall y$. We denote this formula by Ψ. If Ψ is true then Φ is true. Now we show that Ψ is true.
Notice that the formula Ψ has the form $\Psi = \exists x_1 \cdots \exists x_n \forall y \exists x \psi$.

We proceed by an induction on n. For $n = 0$ up to renaming ψ must be the formula $\{y \vee x, \neg y \vee \neg x\}$. Clearly, $\forall y \exists x \psi$ is true. Suppose Ψ has more than one $\exists$–variables. Since $\psi_{|\exists}$ is in *MU(1)* , Ψ has a disjunctive splitting on some variable.

Case 1: ψ has a disjunctive splitting $(\psi^x, \psi^{\neg x})$ on x. Then one of the clauses $C_1' = (\cdots y \vee \cdots)$ and $C_2' = (\cdots \neg y \vee \cdots)$ is in ϕ^x and the other is in $\phi^{\neg x}$.

If we remove the clauses C_1' and C_2' from ψ^x and $\psi^{\neg x}$, then both formulas are satisfiable for the existential part independent of the value of x, because $\psi^x_{|\exists}$ and $\psi^{\neg x}_{|\exists}$ are minimal unsatisfiable (see Lemma 1(2)). Together with $x \leftrightarrow \neg y$, we see that the formula is true.

Case 2: ψ has a disjunctive splitting $(\psi^{x_i}, \psi^{\neg x_i})$ on some $x_i \in \{x_1, \cdots, x_n\}$. Since the two splitting formulas have distinct $\exists$–variables (see Lemma 1), w.l.o.g. we assume that the clauses $(\cdots y \vee L \vee \cdots)$ and $(\cdots \neg y \vee \neg L \vee \cdots)$ are in ψ^{x_i}. The by the induction hypothesis we know that $\exists x_1 \cdots \exists x_{i-1} \forall y \exists x \psi^{x_i}$ is true. Here w.l.o.g. we assume that the $\exists$–variables of ψ^{x_i} are $x_1, \cdots, x_{i-1}, x$. Then for $x_i = 0$ the formula Ψ is true. ∎

Our next task is to prove Lemma 3.

Lemma 3 *Let $\Phi = Q\phi$ be a QCNF formula such that $\phi_{|\exists} \in$ MU(1) . Suppose the case in Lemma 2 does not occur. Then Φ is true if and only if in any splitting tree of ϕ there is a formula ψ such that there is a $\forall$–variable y:*

(1) For all $x \in var(\psi_{|\exists})$, $y < x$.

(2) There are clauses f_1 and f_2 in the two children of ψ, respectively, such that $y \in f_1$ and $\neg y \in f_2$.

Proof: (Outline of the proof) From right to left: Suppose all the $\exists$–variables of Φ are $x_1, \cdots, x_n$, and suppose all the $\exists$–variables in ψ are $x_1, \cdots, x_i$. We remove from ϕ all universal literals except y and $\neg y$. The resulting formula is denoted as ϕ'. Further, w.l.o.g. we assume $x_i = x$. Next we show that $\exists x_{i+1} \cdots \exists x_n \forall y \exists x_1 \cdots \exists x_i \phi'$ is true.

We proceed by an induction on n. If $n = 1$ then clearly the lemma holds. Suppose $n > 1$. If ψ is ϕ' then it is easy to see that $\forall y \exists x_1 \cdots \exists x_n \psi$ is true. Now suppose the splitting formulas of ϕ' are ϕ'^{x_j} and $\phi'^{\neg x_j}$. W.l.o.g., we assume that ψ occurs in the splitting tree of ϕ'^{x_j}. Then by the induction hypothesis we know that $\exists x_{i+1} \cdots \exists x_{j-1} \forall y \exists x_1 \cdots \exists x_i \phi'^{x_j}$ is true. Since the splitting is disjunctive, we can see that formula $\Phi' := \exists x_{i+1} \cdots \exists x_n \forall y \exists x_1 \cdots \exists x_i \phi'$ is true.

For any x_j $(j > i)$, if $y < x_j$ in $Q\phi$ then in Φ' we move $\exists x_j$ to the right of $\forall y$. The resulting formula is denoted as Φ''. Then Φ'' is true. Since up to some removed universal variables Φ equals Φ'', the formula Φ is true.

¿From left to right: We proceed by induction on the number of $\exists$–variables. Suppose Φ has only one $\exists$–variable, say x. Then Φ must be of the form $\forall y_1 \cdots \forall y_i \exists x \forall y_{i+1} \cdots \forall y_k \phi$. Then Φ is false if and only if $\Phi' = \forall y_1 \cdots \forall y_i \exists x \phi'$ is false, where ϕ' is obtained from ϕ by removing all occurrence of y_j and $\neg y_j$, $j > i$. By the assumption, ϕ' contains no occurrences of any complementary

universal literals y_j and $\neg y_j$, $j < i$. Therefore, Φ' is false and since $\phi_{|\exists}$ is in *MU(1)* , we obtain Φ is false.
Suppose ϕ has more than one $\exists$–variables. Suppose in the splitting tree ϕ^x and $\phi^{\neg x}$ are the two splitting formulas of ϕ. Then both $Q^x\phi^x$ and $Q^{\neg x}\phi^{\neg x}$ satisfy the condition of the lemma, where Q^x resp. $Q^{\neg x}$ is obtained from Q by removing quantifiers whose variables do not occur in ϕ^x resp. $\phi^{\neg x}$. Thus, by the induction hypothesis, both $Q^x\phi^x$ and $Q^{\neg x}\phi^{\neg x}$ are false. Again by the assumption of the lemma, we can show that Φ is false. ■

5 Conclusion

We have shown that the minimal falsity problem for *QCNF* formulas with deficiency 1 is solvable in polynomial time. But for fixed deficiency $k \geq 2$ the problem *MF(k)* remains open.

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