An Analysis of Core-guided Maximum Satisfiability Solvers Using Linear Programming

George Katsirelos
INRAE, France
SAT 23

July 11, 2023

Introduction

- MaxSAT solvers:
  - Core-guided
    - In this talk: PMRES, OLL
  - Implicit Hitting Set
- Different performance characteristics
- Seemingly cannot be reconciled

Introduction

- MaxSAT solvers:
    - Core-guided
        - In this talk: PMRES, OLL
    - Implicit Hitting Set
- Different performance characteristics
- Seemingly cannot be reconciled *

MaxSAT

$$\min \ 3x_1 + 4x_2$$

$s.t.$

$$x_1 \vee x_3 \vee \overline{x}_4$$

$$x_2 \vee x_4 \vee \overline{x}_3$$

$$x_4 \vee x_3$$

- H: a SAT formula
- w: a linear objective function
  - Positive weights only
- *Unweighted*: All coefficients in w are unit

Cores and Correction Sets

- Core: a set of literals which cannot be false simultaneously
  - A positive clause over the literals of the objective
- Correction Set: a set of literals which are true together in a solution
- Hitting set duality

Cores and Correction Sets

- Core: a set of literals which cannot be false simultaneously
  - A positive clause over the literals of the objective
- Correction Set: a set of literals which are true together in a solution
- Hitting set duality
- Cores allow us to ignore H and consider *only* the core structure

Implicit Hitting Set

Procedure *IHS(F)*

$\kappa = \emptyset$
while TRUE **do**
  HS $=$ *Minimum* $-$ *HS*(κ)
  if *HS is a correction set* **then**
    return model of *F*
  Extract core *c* from $F \setminus HS$
  $\kappa = \kappa \cup \{c\}$
return model of *F*

Implicit Hitting Set

Procedure *IHS(F)*

$\kappa = \emptyset$

while TRUE **do**

  $HS = \boxed{Minimum - HS(\kappa)}$ ILP

  if *HS is a correction set* **then**

    return model of *F*

  Extract core *c* from $F \setminus HS$

  $\kappa = \kappa \cup \{c\}$

return model of *F*

Implicit Hitting Set

Procedure *IHS(F)*

$\kappa = \emptyset$

while TRUE **do**

  $HS = \boxed{Minimum - HS(\kappa)}$

  if *HS is a correction set* **then**

    return model of *F*

  Extract $\boxed{\text{core } c \text{ from } F \setminus HS}$

  $\kappa = \kappa \cup \{c\}$

return model of *F*

ILP

cores of *F*

Reformulations

Core-Guided Solvers build Reformulations

- A solution-preserving change of H
- $w^i(x) \neq w^{i-1}(x)$ in general
- BUT
  - $w^i = w^{i-1} - c$
  - $w^i(x) \geq 0$
- The constant c is a lower bound

Core-Guided

Procedure *CG*($F = \langle H, w \rangle$)

$lb = 0$

$f_{HS}^{0} = \emptyset$

for *iteration* $i = 1, \ldots$ **do**

$(a, m^i) = (H \cup f_{HS}^{i-1}) \mid_{w=0}$

if $a \neq \emptyset$ **then**

return m

$(f_{HS}^{i}, w^i) = Reformulate(f_{HS}^{i-1}, w^{i-1}, m^i)$

Core-Guided

Procedure $CG(F = \langle H, w \rangle)$

$lb = 0$

$f^0_{HS} = \emptyset$

for *iteration* $i = 1, \ldots$ **do**

$(a, m^i) = (H \cup f^{i-1}_{HS}) \mid_{w=0}$

if $a \neq \emptyset$ **then**

return m

$(f^i_{HS}, w^i) = \boxed{Reformulate(f^{i-1}_{HS}, w^{i-1}, m^i)}$

Formula increases in complexity

Core-Guided

Procedure *CG($F = \langle H, w\rangle$)*

$lb = 0$

$f_{HS}^{0} = \emptyset$

for *iteration* $i = 1, \ldots$ **do**

$(a, \boxed{m^i}) = (H \cup f_{HS}^{i-1}) \mid_{w=0}$

meta

if $a \neq \emptyset$ **then**

return m

Formula increases in complexity

$(f_{HS}^{i}, w^{i}) = \boxed{Reformulate(f_{HS}^{i-1}, w^{i-1}, m^{i})}$

Reformulations (PMRES)

- Objective: $b_1 + b_2 + b_3 + b_4 + b_5$
- Core: b_1, b_2, b_3, b_4

b_5

b_4

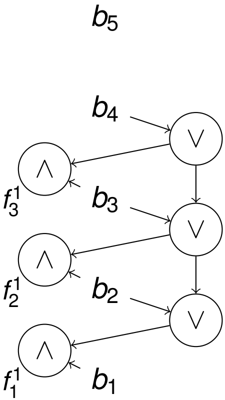
b_3

b_2

b_1

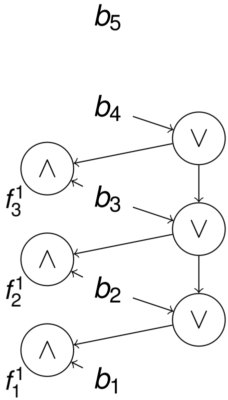
Reformulations (PMRES)

- Objective: $b_1 + b_2 + b_3 + b_4 + b_5$
- Core: b_1, b_2, b_3, b_4



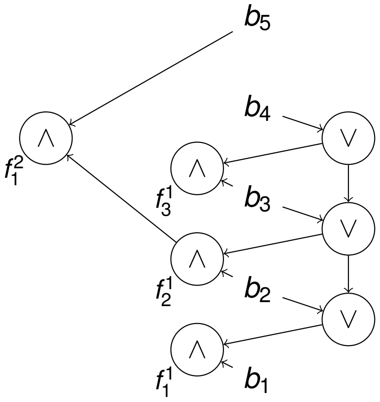
Reformulations (PMRES)

- Objective: $b_5 + f_1^1 + f_2^1 + f_3^1$
- Core: f_2^1, b_5



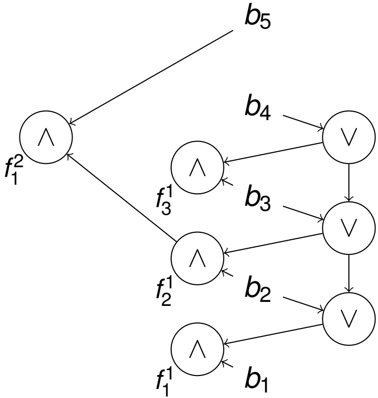
Reformulations (PMRES)

- Objective: $b_5 + f_1^1 + f_2^1 + f_3^1$
- Core: f_2^1, b_5



Reformulations (PMRES)

- Objective: $b_5 + f_1^1 + f_3^1 + f_1^2$



Results

For PMRES, OLL:

1. m^i is logically equivalent to a set of cores of F, $c(m^i)$
2. $\langle f^i_{HS}, w^i \rangle$ encodes the hitting set over $\cup_i c(m^i)$
   - When $f^i_{HS}\,|_{w=0}$ is satisfiable, its solutions are optimum hitting sets
3. $f^i_{HS}\,|_{w=0}$ is satisfiable at every iteration for unweighted instances

   $\Rightarrow$ Optimum hitting set at every iteration

Results

For PMRES, OLL:

1. m^i is logically equivalent to a set of cores of F, $c(m^i)$
2. $\langle f^i_{HS}, w^i \rangle$ encodes the hitting set over $\cup_i c(m^i)$
   - When $f^i_{HS}\,|_{w=0}$ is satisfiable, its solutions are optimum hitting sets
3. $f^i_{HS}\,|_{w=0}$ is satisfiable at every iteration for unweighted instances

   $\Rightarrow$ Optimum hitting set at every iteration

These replicate results already known for PM1/WPM1

Results 2

There exists an ILP that

1. Encodes the hitting set problem over $\cup_i c(m^i)$
2. Has size linear in f^i_{HS}
3. Its linear relaxation has bound at least as great as that computed by PMRES/OLL

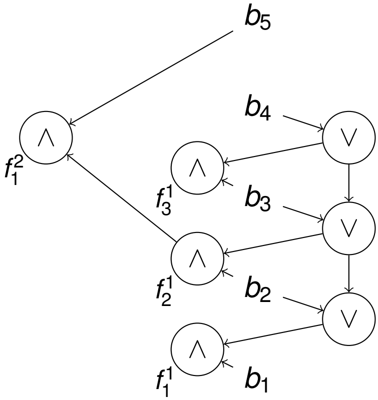
Idea for proof of LP bound

Weighted CSP

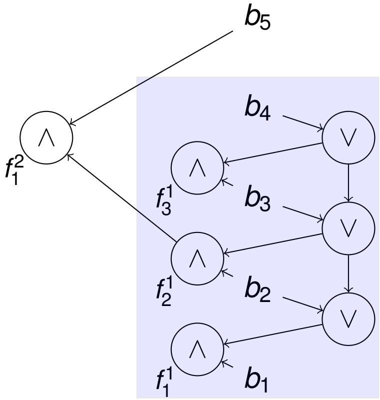
Well known linear relaxation

- Dual solutions define reformulations
- LP optima are arc consistent
    - Not sufficient condition, but enough for this bound
- Show dual solution that replicates PMRES/OLL
    - LP solver will do as well or better

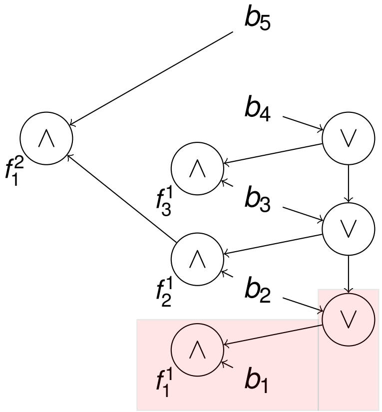
WCSP



WCSP

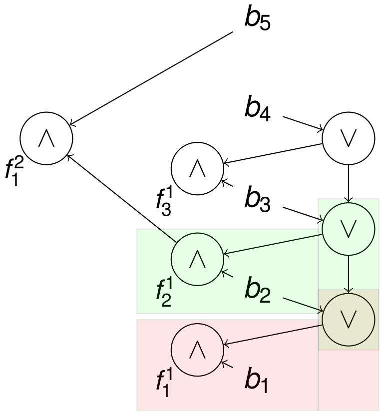


WCSP

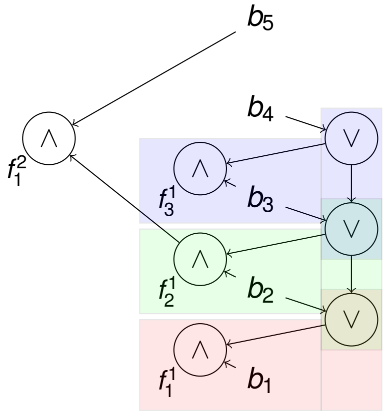


b_5
b_4
$\vee$
$\wedge$
f_1^2
$\wedge$
f_3^1
b_3
$\vee$
$\wedge$
f_2^1
b_2
$\vee$
$\wedge$
f_1^1
b_1

WCSP

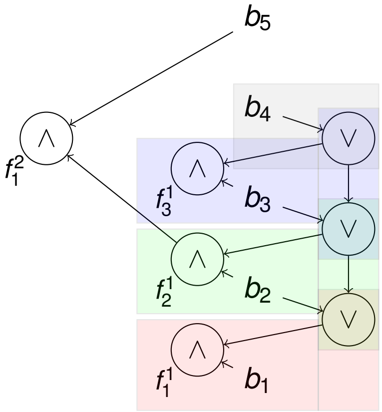


WCSP



b_5
b_4
$\wedge$
f_1^2
$\vee$
$\wedge$
f_3^1
b_3
$\vee$
$\wedge$
f_2^1
b_2
$\vee$
$\wedge$
f_1^1
b_1

WCSP



Conclusions

- Core-guided solvers are hitting set solvers

Conclusions

- OLL and PMRES are also hitting set solvers

Conclusions

- OLL and PMRES are also hitting set solvers

	CG	IHS
cores	Hard	Easy
hitting set	Easy	Hard

Conclusions

- OLL and PMRES are also hitting set solvers

	CG	IHS
cores	Hard	Easy
hitting set	Easy	Hard

- Share progress of core-guided solver via LP

Open problems

Practical

- Design a better algorithm!

Theoretical

- Complexity of $H\mid_{w=0}$ in weighted MaxSAT?
- Complexity of f^i_{hs} in terms of i/other parameters?