

# QCDCL vs QBF Resolution: Further Insights

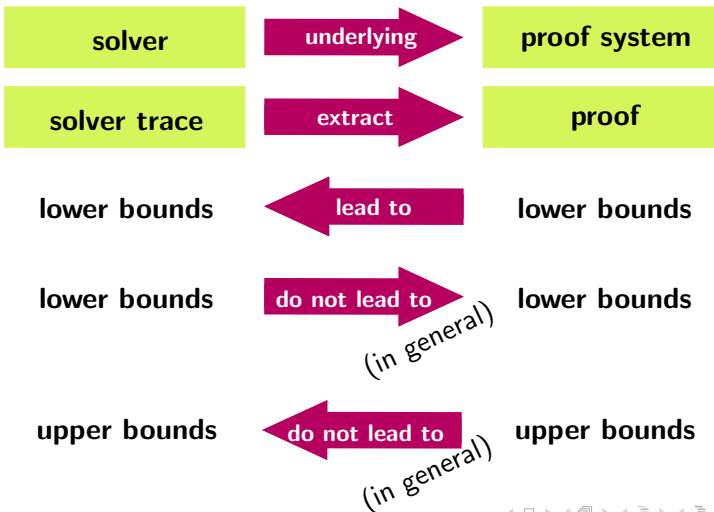
## SAT'23

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# Motivation



# Potentially Redundant Preliminaries - QBF

- QBF = quantified Boolean formulas
- **Input:** formula

$$\Phi = Q_1 x_1 Q_2 x_2 \dots Q_k x_k \cdot \phi(x_1, \dots, x_k)$$

with  $Q_i \in \{\exists, \forall\}$ , variables  $x_1, \dots, x_k$ , and a propositional formula  $\phi$  in conjunctive normal form (CNF).

- **Task:** Find out if  $\Phi$  is true or false.
- **Output:**
  - refutation, if  $\Phi$  is false
  - verification, if  $\Phi$  is true

# Q-Resolution

- in our setting: concentrate on false formulas and refutations
- consider three refutation proof systems:

## Q-Resolution (Q-Res)

- introduced by [Kleine Büning, Karpinski, Flögel 1995]

- extension of Resolution

- Resolution rule: 
$$\frac{x \vee C \qquad \bar{x} \vee D}{C \vee D}$$

for **x existential** and **C ∨ D non-tautological**.

- Reduction rule: 
$$\frac{C \vee u}{C}$$

for **u universal** and there are **no existential literals in C that are quantified right of u**.

- refutation = derivation of empty clause

# Long-distance Q-Resolution and QU-Resolution

## Long-distance Q-Resolution (LDQ-Res)

- introduced by [Zhang, Malik 2002]
- like Q-Resolution, but allows certain resolution steps forbidden in Q-Resolution

## QU-Resolution (QU-Res)

- introduced by [Van Gelder 2012]
- like Q-Resolution, but resolution steps over **universal** literals are now allowed

## Long-distance $QU^+$ -Resolution (LDQU<sup>+</sup>-Res)

- introduced by [Balabanov, Widl, Jiang 2014]
- combines LDQ-Res and QU-Res

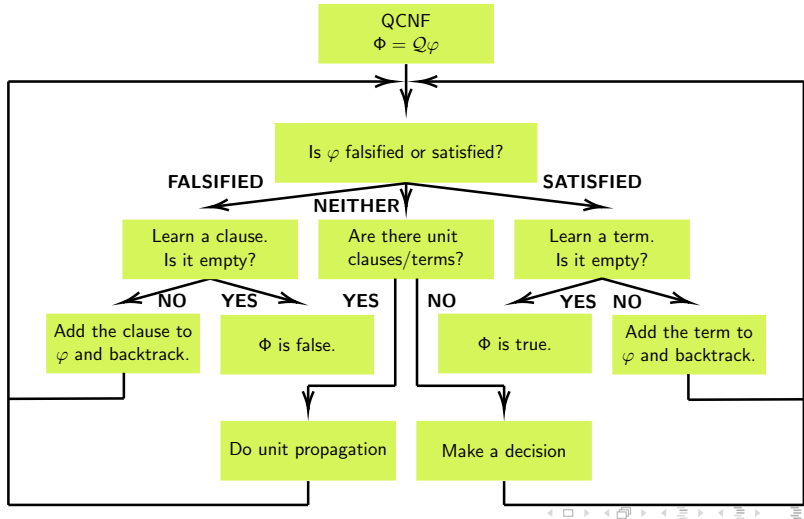
# Obtain Q-Res/LDQ-Res/QU-Res refutations

How can we obtain such a refutation of a given false QBF?

**Answer:** QCDCL = **Q**uantified **C**onflict **D**riven **C**lause **L**earning

- extension of CDCL = **C**onflict **D**riven **C**lause **L**earning
- one of the main paradigms for QBF solving

# (Quantified) Conflict Driven Clause Learning



# Unit propagation in CDCL vs. unit propagation in QCDCL

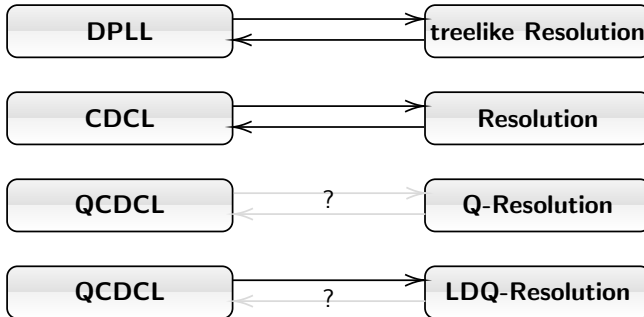
## Unit propagation in CDCL

- If a clause becomes unit under the current assignment, then satisfy this clause.
- **Example:** clause  $C = x \vee y \vee \bar{z}$  and assignment  $\alpha : y \mapsto 1, z \mapsto 0$ , then  $C|_{\alpha} = (x)$   
 $\Rightarrow$ : set  $x \mapsto 1$

## Unit propagation in QCDCL

- If a clause becomes unit under the current assignment **and after reduction**, then satisfy this clause.
- **Example:** clause  $C = x \vee u \vee y$  with prefix  $\exists x \forall u \exists y$  and assignment  $\alpha : y \mapsto 0$ , then  $C|_{\alpha} = (x \vee u)$   
 $\Rightarrow$ : set  $x \mapsto 1$

# (Q)CDCL - What we already (don't) know

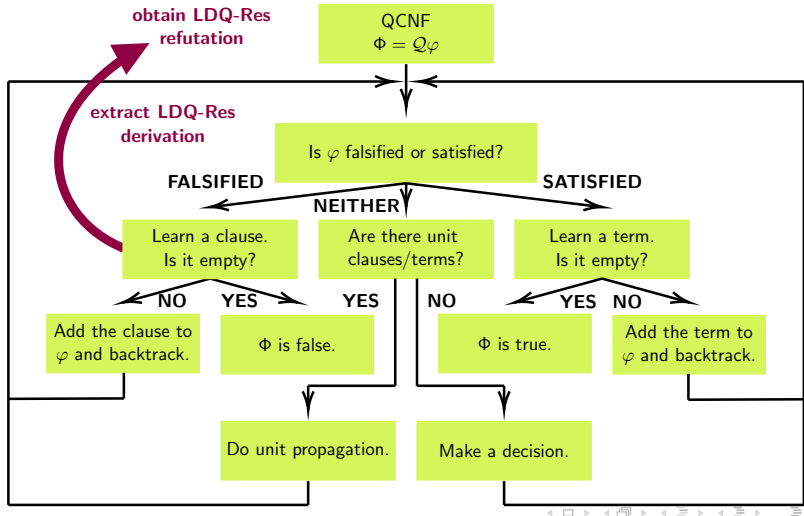


Surprising result:

**Theorem [Pipatsrisawat, Darwiche 2011]**

CDCL (as a proof system) is equivalent to Resolution.

# Extracting proofs from QCDCL



# QBF Solvers and proof systems

What is the connection between QCDCL (as a formalized proof system) and Q-Resolution?

**Theorem [Beyersdorff, B. 2021 (ITCS'21)]**

QCDCL and Q-Resolution are **incomparable**.

There exist exponential separations in both directions.

**However:** There exists a modified variant of QCDCL that is **equivalent** to Q-Resolution.

**New results:**

**Theorem**

There also exist variants of QCDCL that are respectively **equivalent** to QU-Res and a modified version of LDQ-Res.

# QBF Solvers and proof systems

**Question:** What is the weakness of (classical) QCDCL? Why does it not even simulate Q-Resolution?

## Theorem [B., Beyersdorff 2021 (SAT'21)]

There exists a lower bound technique of (classical) QCDCL for a particular class of formulas.

The technique exploits the fact that decisions need to follow quantification order.

# Modifying QCDCL

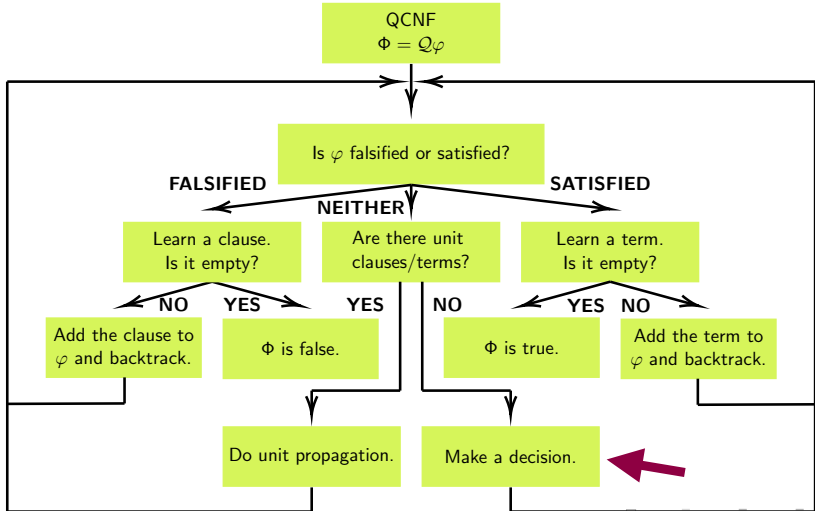
**Aim:** Modify QCDCL such that it can characterise proof systems.

**Idea:** Apply three kinds of policies to QCDCL:

- Decision policies
- Reduction policies
- Propagation policies

**We can define QCDCL variants by choosing one policy from each category.**

# Modifying QCDCL - Decision policies



# Different policies in QCDCL

## ■ How to select **decision** literals

- follow the order of the prefix (practical QCDCL, **LEV-ORD**)
- use arbitrary order (**ANY-ORD**)

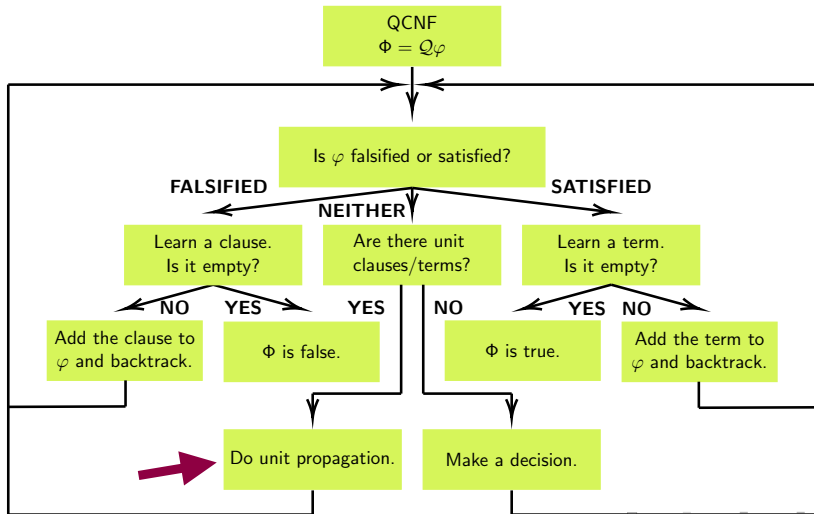
## ■ Whether to use universal reduction in **unit propagation**

- always use universal reduction in unit propagation (practical QCDCL, **ALL-RED**)
- just use plain unit propagation as in CDCL (**NO-RED**)
- choose whether to use universal reduction or not (**ANY-RED**)

## ■ Which variables to propagate

- unit propagation only for existential variables (practical QCDCL, **EXI-PROP**)
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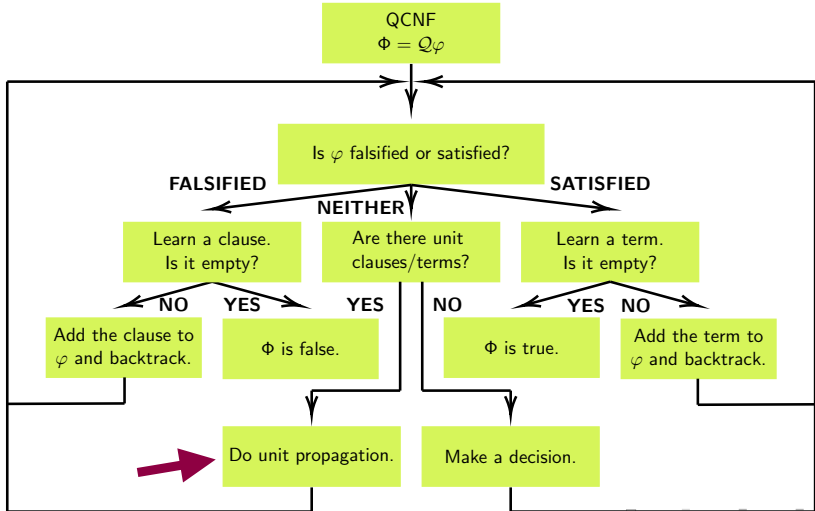
# Modifying QCDCL - Reduction policies



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# Modifying QCDCL - Propagation policies



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## Theorem

All combinations of policies yield sound algorithms.

# Different policies in QCDCL

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**Example:** practical QCDCL = QCDCL<sup>LEV-ORD, ALL-RED, EXI-PROP</sup>

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**Example:** QCDCL<sub>LEV-ORD, NO-RED, ALL-PROP</sub> implemented in [Slivovsky 2022]

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$$\text{Q-Res} \equiv \text{QCDCL}_{\text{NO-RED, EXI-PROP}}^{\text{ANY-ORD}}$$

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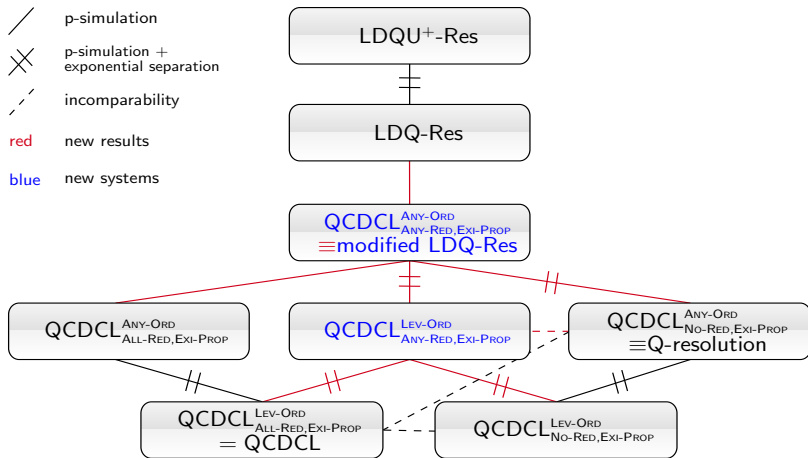
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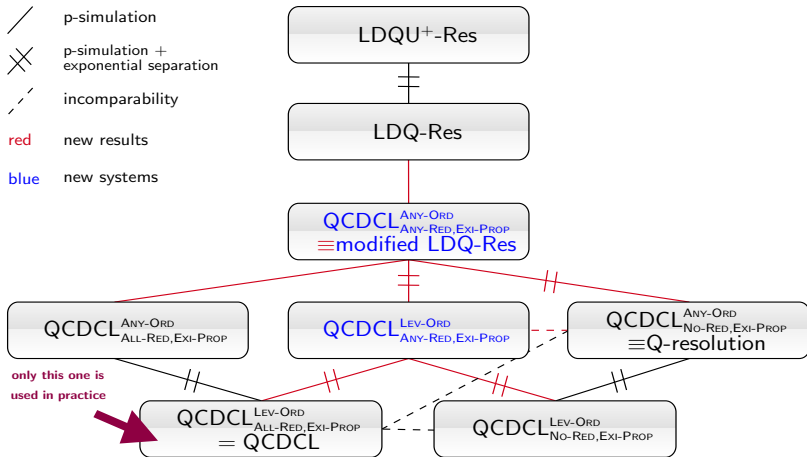
## Theorem

modified LDQ-Res  $\equiv$  QCDCL<sub>ANY-RED, EXI-PROP</sub><sup>ANY-ORD</sup>

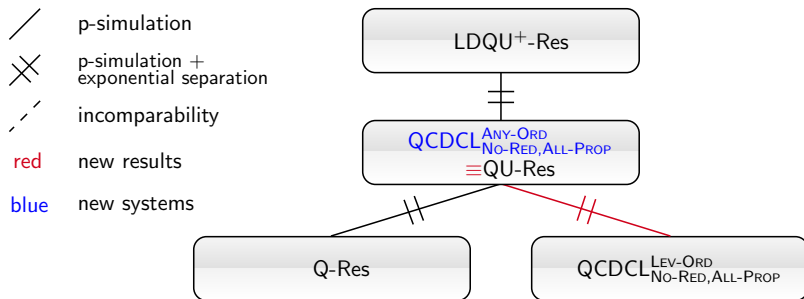
# Overview I



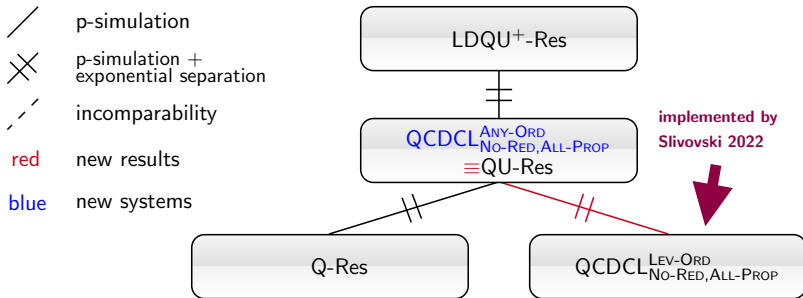
# Overview I



# Overview II



# Overview II



# Fin

**Thanks for listening.**