

Validation of QBF Encodings with Winning Strategies

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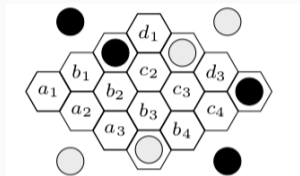
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Validation Challenges

- Is the problem encoded correctly?
- Do encoding variations preserve correctness?

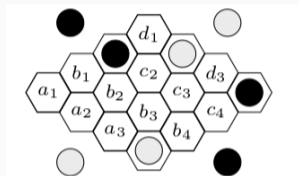
A case study: Hex encoding

Hein Puzzle 12



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Structure

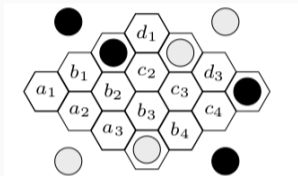
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$$\exists W_1, \dots, W_k$$

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Encoding Variations

- LN : both players can only occupy empty cells

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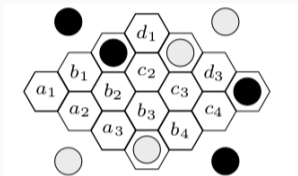
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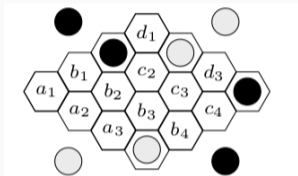
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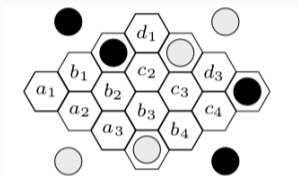
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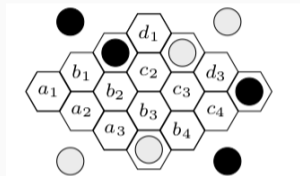
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Move variables

e	1	2	3	0
a	4	5	6	0
e	7	8	9	0
a	10	11	12	0
e	13	14	15	0
a	16	17	18	0
e	19	20	21	0

Validation of QBF Encodings

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- All QBF solvers provide outermost assignment.
- Extract inner variable assignments from modified QBFs

An Outline for Interactive Play

Pseudocode for True instance validation

```
1: for  $i = 1$ ;  $i < \text{max}$ ;  $i = i + 2$  do  
2:   Extract  $\exists$ -move  
3:   Input  $\forall$ -move  
4: end for  
5: Check the Assertion
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$\exists M^1$

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Example Assertions

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Expected Assertion Status for Encoding variations

Inst: / Assert:	GA	LBA
LN-Hein-12	✓	✓
SN-Hein-12	✓	✓
SN-R-Hein-12	✓	✗

Validation results from SQval

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Instance: / Assertion:	Static		Dynamic	
	GA	LBA	GA	LBA
LN-Hein-12	100/0	100/0	100/0	100/0
SN-Hein-12	100/0	100/0	100/0	100/0
SN-R-Hein-12	100/0	91/9	100/0	84/16

No of valid/invalid runs by SQval for Hein-12

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Equivalence Check of Similar Encodings

Equivalence Check of Encodings with Common Winning Strategies

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- Check if the new formula $\phi_2 \wedge S_2$ is also True.
- If False, the encodings are not equivalent.

Results on Equivalence Check for Hein-12

Subsumption ($Q1 \sqsubseteq Q2$) test results from SQval

Q1: / Q2:	LN	SN	SN-R
LN	T	T	T
SN	T	T	T
SN-R	F	F	T

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- SN-R allows overwriting black moves, whereas LN and SN does not.

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- SN-R allows overwriting black moves, whereas LN and SN does not.
- Each subsumption checks take a few seconds at most.

Scaling Validation

Scalable Validation with Partial Certificates

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- Each validation iteration needs to solve a QBF instance

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- For rest of the layers, we interactively play with a QBF solver
- Partial certificates are orders of magnitude more compact.
- Solving QBF formulas with fewer quantifiers is several orders faster.

Validating Hein-09 with Partial Certificates

A Hein-09 instance:

- open positions : 10
- moves : 9
- solving time : 1203 seconds
- certificate size : 7.8GB

Validating Hein-09 with Partial Certificates

Certificate sizes for Hein-09 instances

Instance	Full Certificate	Partial Certificate				
		k=1	k=3	k=5	k=7	k=9
LN-Hein-09	7.8G	108	1.1K	21.8K	434K	8.2M
SN-Hein-09	641.6M	96	1.1K	23.4K	437.3K	8.2M
SN-R-Hein-09	532.7M	96	1.2K	24.2K	461.1K	8.7M

Size in Bytes

Validating Hein-09 with Partial Certificates

Validating Goal Assertion (GA)

	static		dynamic		hybrid-L3	
Instance	PM	TT	PM	TT	PM	TT
LN-Hein-09	-	TO	64.1	1203	1.54	1.6
SN-Hein-09	20.08K	2100	1.54	9.4	1.53	0.4
SN-R-Hein-09	18.35K	1226	1.53	5.4	1.53	0.4

Peak Memory (PM) in MB and Time Taken (TT) in seconds

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	LN-Hein-09		SN-Hein-09		SN-R-Hein-09	
Inst:	Full	L9	Full	L9	Full	L9
LN-Hein-09	-	1.53	-	1.57	-	1.57
SN-Hein-09	63.7	1.53	63.7	1.6	63.7	1.6
SN-R-Hein-09	50.74	1.6	50.78	1.6	50.8	1.69

Peak Memory for subsumption check in GB

Conclusion

Summary

- Validation using interactive play + assertions



SQval

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SQval

Future Work

- Partial traces from QBF solver would help for scalable validation

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Future Work

- Partial traces from QBF solver would help for scalable validation
- Allowing subsumption check beyond encodings with common variables