On CNF Conversion for Disjoint SAT Enumeration

Gabriele Masina, Giuseppe Spallitta, Roberto Sebastiani

UNIVERSITY OF TRENTO

The 26th International Conference on Theory and Applications of Satisfiability Testing

July 5, 2023

Outline

1. Preliminaries
2. Problems of CNF-ization for AllSAT
3. Our solution
4. Experimental results
5. Conclusions & Future Work

Outline

1. Preliminaries
2. Problems of CNF-ization for AllSAT
3. Our solution
4. Experimental results
5. Conclusions & Future Work

SAT enumeration (AllSAT)

The AllSAT problem

AllSAT is the task of enumerating all the models of a Propositional formula φ:

- a set of *total* models $\mathcal{TTA}(\varphi)$
- a set of *partial* models $\mathcal{TA}(\varphi)$ s.t.:
  (a) $\mu \models \varphi$ for all $\mu \in \mathcal{TA}(\varphi)$
    - with μ partial, by "$\mu \models \varphi$" we mean "$\varphi|_\mu = \top$" [1] [2]

  (b) every $\eta \in \mathcal{TTA}(\varphi)$ is a super-assignment of some $\mu \in \mathcal{TA}(\varphi)$

[1] R. Sebastiani (2020). *Are You Satisfied by This Partial Assignment?* arXiv: 2003.04225

[2] S. Möhle, R. Sebastiani, and A. Biere (2020). "Four Flavors of Entailment". In: SAT

SAT enumeration (AllSAT)

The AllSAT problem

AllSAT is the task of enumerating all the models of a Propositional formula φ:

- a set of *total* models $\mathcal{TTA}(\varphi)$
- a set of *partial* models $\mathcal{TA}(\varphi)$ s.t.:
  (a) $\mu \models \varphi$ for all $\mu \in \mathcal{TA}(\varphi)$
    - with μ partial, by "$\mu \models \varphi$" we mean "$\varphi|_\mu = \top$" [1] [2]

  (b) every $\eta \in \mathcal{TTA}(\varphi)$ is a super-assignment of some $\mu \in \mathcal{TA}(\varphi)$

A partial model μ represents 2^k total models, where $k = \#$ of unassigned atoms in μ
$\Longrightarrow$ we want to enumerate partial models that are as short as possible (possibly *minimal*)

[1] R. Sebastiani (2020). *Are You Satisfied by This Partial Assignment?* arXiv: 2003.04225
[2] S. Möhle, R. Sebastiani, and A. Biere (2020). "Four Flavors of Entailment". In: SAT

SAT enumeration (AllSAT)

The *disjoint* AllSAT problem

Disjoint AllSAT is the task of enumerating all the models of a Propositional formula φ:

- a set of *total* models $\mathcal{TTA}(\varphi)$
- a set of disjoint *partial* models $\mathcal{TA}(\varphi)$ s.t.:
  (a) $\mu \models \varphi$ for all $\mu \in \mathcal{TA}(\varphi)$
    - with μ partial, by "$\mu \models \varphi$" we mean "$\varphi|_\mu = \top$" [1] [2]
  (b) every $\eta \in \mathcal{TTA}(\varphi)$ is a super-assignment of some $\mu \in \mathcal{TA}(\varphi)$
  (c) every pair $\mu_i, \mu_j \in \mathcal{TA}(\varphi)$ assigns opposite truth values to at least one atom

A partial model μ represents 2^k total models, where $k = \#$ of unassigned atoms in μ
$\Longrightarrow$ we want to enumerate partial models that are as short as possible (possibly *minimal*)

Disjointness needed in many applications like #SMT, Weighted Model Integration (WMI)

[1] R. Sebastiani (2020). *Are You Satisfied by This Partial Assignment?* arXiv: 2003.04225
[2] S. Möhle, R. Sebastiani, and A. Biere (2020). "Four Flavors of Entailment". In: SAT

Projected AllSAT

The Projected AllSAT problem

Given $\varphi(\mathbf{A} \cup \mathbf{B})$ s.t. $\mathbf{A} \cap \mathbf{B} = \emptyset$ and $\mathbf{A}$ is a set of *relevant atoms*.
Enumerate the models of φ *projected* over $\mathbf{A}$, i.e. $\mathcal{TA}(\exists \mathbf{B}.\varphi(\mathbf{A} \cup \mathbf{B}))$.

Projected AllSAT

The *disjoint* Projected AllSAT problem

Given $\varphi(\mathbf{A} \cup \mathbf{B})$ s.t. $\mathbf{A} \cap \mathbf{B} = \emptyset$ and $\mathbf{A}$ is a set of *relevant atoms*.
Enumerate the disjoint models of φ *projected* over $\mathbf{A}$, i.e. $\mathcal{TA}(\exists \mathbf{B}.\varphi(\mathbf{A} \cup \mathbf{B}))$.

Projected AllSAT

The *disjoint* Projected AllSAT problem

Given $\varphi(\mathbf{A} \cup \mathbf{B})$ s.t. $\mathbf{A} \cap \mathbf{B} = \emptyset$ and $\mathbf{A}$ is a set of *relevant atoms*.
Enumerate the disjoint models of φ *projected* over $\mathbf{A}$, i.e. $\mathcal{TA}(\exists \mathbf{B}.\varphi(\mathbf{A} \cup \mathbf{B}))$.

Minimal disjoint Projected AllSAT in practice

Enumerate partial truth assignments $\mu_1, \dots, \mu_i, \dots \mu_N$, where each $\mu_i \stackrel{\text{def}}{=} \mu_i^{\mathbf{A}} \cup \eta_i^{\mathbf{B}}$ is s.t.[a]:

(i) (*satisfiability*) $\mu_i \models \varphi$

(ii) (*disjointness*) for each $j < i$, $\mu_i^{\mathbf{A}}, \mu_j^{\mathbf{A}}$ assign opposite truth values to some atom in $\mathbf{A}$

(iii) (*minimality*) $\mu_i^{\mathbf{A}}$ is *minimal*: no literal can be dropped from it without losing properties (i) and (ii)

$\Longrightarrow$ return $\{\mu_i^{\mathbf{A}}\}_{i=1}^{N}$

[a] $\mu_i^{\mathbf{A}}$ is a *partial* assignment over $\mathbf{A}$ and $\eta_i^{\mathbf{B}}$ is a *total* assignment over $\mathbf{B}$

What about non-CNF formulas?

Outline

Tseitin encoding

Consider:

- a Propositional formula $\varphi(\mathbf{A})$
- its Tseitin encoding $\mathsf{CNF}_{\mathsf{Ts}}(\varphi)(\mathbf{A} \cup \mathbf{B})$ where $\mathbf{B}$ is the set of *fresh* labels
- **Intuition**: $\mathsf{CNF}_{\mathsf{Ts}}(\varphi)$ recursively rewritten as $\mathsf{CNF}_{\mathsf{Ts}}(\varphi[\varphi_i|B_i]) \wedge \mathsf{CNF}_{\mathsf{Ts}}(B_i \leftrightarrow \varphi_i)$, φ_i subformula, B_i fresh

Tseitin encoding

Consider:

- a Propositional formula $\varphi(\mathbf{A})$
- its Tseitin encoding $\mathsf{CNF}_{\mathsf{Ts}}(\varphi)(\mathbf{A} \cup \mathbf{B})$ where $\mathbf{B}$ is the set of *fresh* labels
- **Intuition**: $\mathsf{CNF}_{\mathsf{Ts}}(\varphi)$ recursively rewritten as $\mathsf{CNF}_{\mathsf{Ts}}(\varphi[\varphi_i|B_i]) \wedge \mathsf{CNF}_{\mathsf{Ts}}(B_i \leftrightarrow \varphi_i)$, φ_i subformula, B_i fresh

Then

- $\varphi(\mathbf{A}) \equiv \exists \mathbf{B}.\mathsf{CNF}_{\mathsf{Ts}}(\varphi)(\mathbf{A} \cup \mathbf{B})$
- $\mathcal{TA}(\varphi(\mathbf{A}))$ can be enumerated as $\mathcal{TA}(\exists \mathbf{B}.\mathsf{CNF}_{\mathsf{Ts}}(\varphi)(\mathbf{A} \cup \mathbf{B}))$
  $\Longrightarrow$ project the models of $\mathsf{CNF}_{\mathsf{Ts}}(\varphi)$ onto $\mathbf{A}$ only

Let $\varphi \stackrel{\text{def}}{=} \overbrace{(A_1 \land A_2)}^{B_1} \lor \overbrace{(\overbrace{(\overbrace{(A_3 \lor A_4)}^{B_2} \land \overbrace{(A_5 \lor A_6)}^{B_3})}^{B_4} \leftrightarrow A_7)}^{B_5}$

$$\text{Let } \varphi \stackrel{\text{def}}{=} \overbrace{(A_1 \wedge A_2)}^{B_1} \vee \overbrace{(\, \overbrace{(\, \overbrace{(A_3 \vee A_4)}^{B_2} \wedge \overbrace{(A_5 \vee A_6)}^{B_3} \,)}^{B_4} \leftrightarrow A_7)}^{B_5}$$

$$\begin{array}{rll}
\mathsf{CNF}_{\mathsf{Ts}}(\varphi) \stackrel{\text{def}}{=} & (\neg B_1 \vee A_1) \wedge (\neg B_1 \vee A_2) \wedge (B_1 \vee \neg A_1 \vee \neg A_2) \quad \wedge & //(B_1 \leftrightarrow (A_1 \wedge A_2)) \\
& (B_2 \vee \neg A_3) \wedge (B_2 \vee \neg A_4) \wedge (\neg B_2 \vee A_3 \vee A_4) \quad \wedge & //(B_2 \leftrightarrow (A_3 \vee A_4)) \\
& (B_3 \vee \neg A_5) \wedge (B_3 \vee \neg A_6) \wedge (\neg B_3 \vee A_5 \vee A_6) \quad \wedge & //(B_3 \leftrightarrow (A_5 \vee A_6)) \\
& (\neg B_4 \vee B_2) \wedge (\neg B_4 \vee B_3) \wedge (B_4 \vee \neg B_2 \vee \neg B_3) \quad \wedge & //(B_4 \leftrightarrow (B_2 \wedge B_3)) \\
& (\neg B_5 \vee B_4 \vee \neg A_7) \wedge (\neg B_5 \vee \neg B_4 \vee A_7) \quad \wedge & //(B_5 \leftrightarrow (B_4 \leftrightarrow A_7)) \\
& (B_5 \vee B_4 \vee A_7) \wedge (B_5 \vee \neg B_4 \vee \neg A_7) \quad \wedge & \\
& (B_1 \vee B_5) &
\end{array}$$

Let $\varphi \stackrel{\text{def}}{=} \underbrace{(A_1 \wedge A_2)}_{B_1} \vee \underbrace{(\underbrace{(\underbrace{(A_3 \vee A_4)}_{B_2} \wedge \underbrace{(A_5 \vee A_6)}_{B_3})}_{B_4} \leftrightarrow A_7)}_{B_5}$

$$
\begin{array}{lllll}
\mathsf{CNF_{Ts}}(\varphi) \stackrel{\text{def}}{=} & (\neg B_1 \vee A_1) \wedge (\neg B_1 \vee A_2) \wedge (B_1 \vee \neg A_1 \vee \neg A_2) & \wedge & //(B_1 \leftrightarrow (A_1 \wedge A_2)) \\
& (B_2 \vee \neg A_3) \wedge (B_2 \vee \neg A_4) \wedge (\neg B_2 \vee A_3 \vee A_4) & \wedge & //(B_2 \leftrightarrow (A_3 \vee A_4)) \\
& (B_3 \vee \neg A_5) \wedge (B_3 \vee \neg A_6) \wedge (\neg B_3 \vee A_5 \vee A_6) & \wedge & //(B_3 \leftrightarrow (A_5 \vee A_6)) \\
& (\neg B_4 \vee B_2) \wedge (\neg B_4 \vee B_3) \wedge (B_4 \vee \neg B_2 \vee \neg B_3) & \wedge & //(B_4 \leftrightarrow (B_2 \wedge B_3)) \\
& (\neg B_5 \vee B_4 \vee \neg A_7) \wedge (\neg B_5 \vee \neg B_4 \vee A_7) & \wedge & //(B_5 \leftrightarrow (B_4 \leftrightarrow A_7)) \\
& (B_5 \vee B_4 \vee A_7) \wedge (B_5 \vee \neg B_4 \vee \neg A_7) & \wedge & \\
& (B_1 \vee B_5) & &
\end{array}
$$

Problem

Consider $\mu^{\mathbf{A}} \stackrel{\text{def}}{=} \{\neg A_3, \neg A_4, \neg A_7\}$ s.t. $\mu^{\mathbf{A}} \models \varphi$.

Let $\varphi \stackrel{\text{def}}{=} \overbrace{(A_1 \wedge A_2)}^{B_1} \vee \overbrace{(\overbrace{(\overbrace{(A_3 \vee A_4)}^{B_2} \wedge \overbrace{(A_5 \vee A_6)}^{B_3})}^{B_4} \leftrightarrow A_7)}^{B_5}$

$$
\begin{aligned}
\mathsf{CNF}_{\mathsf{Ts}}(\varphi) \stackrel{\text{def}}{=} & (\neg B_1 \vee A_1) \wedge (\neg B_1 \vee A_2) \wedge (B_1 \vee \neg A_1 \vee \neg A_2) && \wedge \quad //(B_1 \leftrightarrow (A_1 \wedge A_2)) \\
& (B_2 \vee \neg A_3) \wedge (B_2 \vee \neg A_4) \wedge (\neg B_2 \vee A_3 \vee A_4) && \wedge \quad //(B_2 \leftrightarrow (A_3 \vee A_4)) \\
& (B_3 \vee \neg A_5) \wedge (B_3 \vee \neg A_6) \wedge (\neg B_3 \vee A_5 \vee A_6) && \wedge \quad //(B_3 \leftrightarrow (A_5 \vee A_6)) \\
& (\neg B_4 \vee B_2) \wedge (\neg B_4 \vee B_3) \wedge (B_4 \vee \neg B_2 \vee \neg B_3) && \wedge \quad //(B_4 \leftrightarrow (B_2 \wedge B_3)) \\
& (\neg B_5 \vee B_4 \vee \neg A_7) \wedge (\neg B_5 \vee \neg B_4 \vee A_7) && \wedge \quad //(B_5 \leftrightarrow (B_4 \leftrightarrow A_7)) \\
& (B_5 \vee B_4 \vee A_7) \wedge (B_5 \vee \neg B_4 \vee \neg A_7) && \wedge \\
& (B_1 \vee B_5)
\end{aligned}
$$

Problem

Consider $\mu^{\mathbf{A}} \stackrel{\text{def}}{=} \{\neg A_3, \neg A_4, \neg A_7\}$ s.t. $\mu^{\mathbf{A}} \models \varphi$.

$\mu^{\mathbf{A}} \not\models \exists \mathbf{B}.\mathsf{CNF}_{\mathsf{Ts}}(\varphi)!$ i.e., $\nexists \eta^{\mathbf{B}}$ s.t. $(\mu^{\mathbf{A}} \cup \eta^{\mathbf{B}} \models \mathsf{CNF}_{\mathsf{Ts}}(\varphi))$

Let $\varphi \stackrel{\text{def}}{=} \underbrace{(A_1 \wedge A_2)}^{B_1} \vee \overbrace{(\overbrace{(\overbrace{(A_3 \vee A_4)}^{B_2} \wedge \overbrace{(A_5 \vee A_6)}^{B_3})}^{B_4} \leftrightarrow A_7)}^{B_5}$

$$
\begin{array}{lll}
\mathsf{CNF_{Ts}}(\varphi) \stackrel{\text{def}}{=} & (\neg B_1 \vee A_1) \wedge (\neg B_1 \vee A_2) \wedge (B_1 \vee \neg A_1 \vee \neg A_2) & \wedge \quad //(B_1 \leftrightarrow (A_1 \wedge A_2)) \\
& (B_2 \vee \neg A_3) \wedge (B_2 \vee \neg A_4) \wedge (\neg B_2 \vee A_3 \vee A_4) & \wedge \quad //(B_2 \leftrightarrow (A_3 \vee A_4)) \\
& (B_3 \vee \neg A_5) \wedge (B_3 \vee \neg A_6) \wedge (\neg B_3 \vee A_5 \vee A_6) & \wedge \quad //(B_3 \leftrightarrow (A_5 \vee A_6)) \\
& (\neg B_4 \vee B_2) \wedge (\neg B_4 \vee B_3) \wedge (B_4 \vee \neg B_2 \vee \neg B_3) & \wedge \quad //(B_4 \leftrightarrow (B_2 \wedge B_3)) \\
& (\neg B_5 \vee B_4 \vee \neg A_7) \wedge (\neg B_5 \vee \neg B_4 \vee A_7) & \wedge \quad //(B_5 \leftrightarrow (B_4 \leftrightarrow A_7)) \\
& (B_5 \vee B_4 \vee A_7) \wedge (B_5 \vee \neg B_4 \vee \neg A_7) & \wedge \\
& (B_1 \vee B_5) &
\end{array}
$$

Problem

Consider $\mu^{\mathbf{A}} \stackrel{\text{def}}{=} \{\neg A_3, \neg A_4, \neg A_7\}$ s.t. $\mu^{\mathbf{A}} \models \varphi$.

$\mu^{\mathbf{A}} \not\models \exists \mathbf{B}.\mathsf{CNF_{Ts}}(\varphi)!$ i.e., $\nexists \eta^{\mathbf{B}}$ s.t. $(\mu^{\mathbf{A}} \cup \eta^{\mathbf{B}} \models \mathsf{CNF_{Ts}}(\varphi))$

E.g., $\eta^{\mathbf{B}} \stackrel{\text{def}}{=} \{\neg B_2, \ldots\}$

Let $\varphi \stackrel{\text{def}}{=} \overset{B_1}{(A_1 \wedge A_2)} \vee \overset{B_5}{\left(\overset{B_4}{\left(\overset{B_2}{(A_3 \vee A_4)} \wedge \overset{B_3}{(A_5 \vee A_6)} \right)} \leftrightarrow A_7 \right)}$

$$
\begin{array}{lll}
\mathsf{CNF}_{\mathsf{Ts}}(\varphi) \stackrel{\text{def}}{=} & (\neg B_1 \vee A_1) \wedge (\neg B_1 \vee A_2) \wedge (B_1 \vee \neg A_1 \vee \neg A_2) \quad \wedge & //(B_1 \leftrightarrow (A_1 \wedge A_2)) \\
& (B_2 \vee \neg A_3) \wedge (B_2 \vee \neg A_4) \wedge (\neg B_2 \vee A_3 \vee A_4) \quad \wedge & //(B_2 \leftrightarrow (A_3 \vee A_4)) \\
& (B_3 \vee \neg A_5) \wedge (B_3 \vee \neg A_6) \wedge (\neg B_3 \vee A_5 \vee A_6) \quad \wedge & //(B_3 \leftrightarrow (A_5 \vee A_6)) \\
& (\neg B_4 \vee B_2) \wedge (\neg B_4 \vee B_3) \wedge (B_4 \vee \neg B_2 \vee \neg B_3) \quad \wedge & //(B_4 \leftrightarrow (B_2 \wedge B_3)) \\
& (\neg B_5 \vee B_4 \vee \neg A_7) \wedge (\neg B_5 \vee \neg B_4 \vee A_7) \quad \wedge & //(B_5 \leftrightarrow (B_4 \leftrightarrow A_7)) \\
& (B_5 \vee B_4 \vee A_7) \wedge (B_5 \vee \neg B_4 \vee \neg A_7) \quad \wedge & \\
& (B_1 \vee B_5) &
\end{array}
$$

Problem

Consider $\mu^{\mathbf{A}} \stackrel{\text{def}}{=} \{\neg A_3, \neg A_4, \neg A_7\}$ s.t. $\mu^{\mathbf{A}} \models \varphi$.

$\mu^{\mathbf{A}} \not\models \exists \mathbf{B}.\mathsf{CNF}_{\mathsf{Ts}}(\varphi)!$ i.e., $\nexists \eta^{\mathbf{B}}$ s.t. $(\mu^{\mathbf{A}} \cup \eta^{\mathbf{B}} \models \mathsf{CNF}_{\mathsf{Ts}}(\varphi))$

E.g., $\eta^{\mathbf{B}} \stackrel{\text{def}}{=} \{\neg B_2, \neg B_4, \ldots\}$

Let $\varphi \stackrel{\text{def}}{=} \underbrace{(A_1 \wedge A_2)}_{B_1} \vee ((\underbrace{(A_3 \vee A_4)}_{B_2} \wedge \underbrace{(A_5 \vee A_6)}_{B_3})_{B_4} \leftrightarrow A_7)_{B_5}$

$$
\begin{array}{lllll}
\mathsf{CNF_{Ts}}(\varphi) \stackrel{\text{def}}{=} & (\neg B_1 \vee A_1) \wedge (\neg B_1 \vee A_2) \wedge (B_1 \vee \neg A_1 \vee \neg A_2) & \wedge & //(B_1 \leftrightarrow (A_1 \wedge A_2)) \\
& (B_2 \vee \neg A_3) \wedge (B_2 \vee \neg A_4) \wedge (\neg B_2 \vee A_3 \vee A_4) & \wedge & //(B_2 \leftrightarrow (A_3 \vee A_4)) \\
& (B_3 \vee \neg A_5) \wedge (B_3 \vee \neg A_6) \wedge (\neg B_3 \vee A_5 \vee A_6) & \wedge & //(B_3 \leftrightarrow (A_5 \vee A_6)) \\
& (\neg B_4 \vee B_2) \wedge (\neg B_4 \vee B_3) \wedge (B_4 \vee \neg B_2 \vee \neg B_3) & \wedge & //(B_4 \leftrightarrow (B_2 \wedge B_3)) \\
& (\neg B_5 \vee B_4 \vee \neg A_7) \wedge (\neg B_5 \vee \neg B_4 \vee A_7) & \wedge & //(B_5 \leftrightarrow (B_4 \leftrightarrow A_7)) \\
& (B_5 \vee B_4 \vee A_7) \wedge (B_5 \vee \neg B_4 \vee \neg A_7) & \wedge & \\
& (B_1 \vee B_5) & &
\end{array}
$$

Problem

Consider $\mu^{\mathbf{A}} \stackrel{\text{def}}{=} \{\neg A_3, \neg A_4, \neg A_7\}$ s.t. $\mu^{\mathbf{A}} \models \varphi$.

$$\mu^{\mathbf{A}} \not\models \exists \mathbf{B}.\mathsf{CNF_{Ts}}(\varphi)! \qquad \text{i.e., } \nexists \eta^{\mathbf{B}} \text{ s.t. } (\mu^{\mathbf{A}} \cup \eta^{\mathbf{B}} \models \mathsf{CNF_{Ts}}(\varphi))$$

E.g., $\eta^{\mathbf{B}} \stackrel{\text{def}}{=} \{\neg B_2, \neg B_4, B_5, \ldots\}$

Let $\varphi \stackrel{\text{def}}{=} \underbrace{(A_1 \wedge A_2)}_{B_1} \vee ((\underbrace{(A_3 \vee A_4)}_{B_2} \wedge \underbrace{(A_5 \vee A_6)}_{B_3})_{B_4} \leftrightarrow A_7)_{B_5}$

$$
\begin{array}{llr}
\mathsf{CNF_{Ts}}(\varphi) \stackrel{\text{def}}{=} & (\neg B_1 \vee A_1) \wedge (\neg B_1 \vee A_2) \wedge (B_1 \vee \neg A_1 \vee \neg A_2) \quad \wedge & //(B_1 \leftrightarrow (A_1 \wedge A_2)) \\
& (B_2 \vee \neg A_3) \wedge (B_2 \vee \neg A_4) \wedge (\neg B_2 \vee A_3 \vee A_4) \quad \wedge & //(B_2 \leftrightarrow (A_3 \vee A_4)) \\
& (B_3 \vee \neg A_5) \wedge (B_3 \vee \neg A_6) \wedge (\neg B_3 \vee A_5 \vee A_6) \quad \wedge & //(B_3 \leftrightarrow (A_5 \vee A_6)) \\
& (\neg B_4 \vee B_2) \wedge (\neg B_4 \vee B_3) \wedge (B_4 \vee \neg B_2 \vee \neg B_3) \quad \wedge & //(B_4 \leftrightarrow (B_2 \wedge B_3)) \\
& (\neg B_5 \vee B_4 \vee \neg A_7) \wedge (\neg B_5 \vee \neg B_4 \vee A_7) \quad \wedge & //(B_5 \leftrightarrow (B_4 \leftrightarrow A_7)) \\
& (B_5 \vee B_4 \vee A_7) \wedge (B_5 \vee \neg B_4 \vee \neg A_7) \quad \wedge & \\
& (B_1 \vee B_5) &
\end{array}
$$

Problem

Consider $\mu^{\mathbf{A}} \stackrel{\text{def}}{=} \{\neg A_3, \neg A_4, \neg A_7\}$ s.t. $\mu^{\mathbf{A}} \models \varphi$.

$\mu^{\mathbf{A}} \not\models \exists \mathbf{B}.\mathsf{CNF_{Ts}}(\varphi)$! i.e., $\nexists \eta^{\mathbf{B}}$ s.t. $(\mu^{\mathbf{A}} \cup \eta^{\mathbf{B}} \models \mathsf{CNF_{Ts}}(\varphi))$

E.g., $\eta^{\mathbf{B}} \stackrel{\text{def}}{=} \{\neg B_2, \neg B_4, B_5, \neg B_1, \neg B_3\} \Longrightarrow (\neg A_1 \vee \neg A_2)$ and $(\neg A_5 \wedge \neg A_6)$ are not satisfied

The impact of Tseitin encoding

The problem

$$\mu^{\mathbf{A}} \models \varphi(\mathbf{A}) \;\not\Longrightarrow\; \mu^{\mathbf{A}} \models \exists \mathbf{B}.\mathsf{CNF}_{\mathsf{Ts}}(\varphi)$$

i.e., it **does not** imply that some $\eta^{\mathbf{B}}$ exists s.t. $\mu^{\mathbf{A}} \cup \eta^{\mathbf{B}} \models \mathsf{CNF}_{\mathsf{Ts}}(\varphi)$.

The impact of Tseitin encoding

The problem

$$\mu^{\mathbf{A}} \models \varphi(\mathbf{A}) \not\Longrightarrow \mu^{\mathbf{A}} \models \exists \mathbf{B}.\mathsf{CNF}_{\mathsf{Ts}}(\varphi)$$

i.e., it **does not** imply that some $\eta^{\mathbf{B}}$ exists s.t. $\mu^{\mathbf{A}} \cup \eta^{\mathbf{B}} \models \mathsf{CNF}_{\mathsf{Ts}}(\varphi)$.

Intuition

Each $(B_i \leftrightarrow \varphi_i)$ **forces** every partial model $\mu^{\mathbf{A}\prime}$ of $\exists \mathbf{B}.\mathsf{CNF}_{\mathsf{Ts}}(\varphi)$ **to assign a truth value to** φ_i (and thus to some of its atoms)

The impact of Tseitin encoding

The problem

$$\mu^{\mathbf{A}} \models \varphi(\mathbf{A}) \;\not\Rightarrow\; \mu^{\mathbf{A}} \models \exists \mathbf{B}.\mathsf{CNF}_{\mathsf{Ts}}(\varphi)$$

i.e., it **does not** imply that some $\eta^{\mathbf{B}}$ exists s.t. $\mu^{\mathbf{A}} \cup \eta^{\mathbf{B}} \models \mathsf{CNF}_{\mathsf{Ts}}(\varphi)$.

Intuition

Each $(\boldsymbol{B_i \leftrightarrow \varphi_i})$ **forces** every partial model $\mu^{\mathbf{A}'}$ of $\exists \mathbf{B}.\mathsf{CNF}_{\mathsf{Ts}}(\varphi)$ **to assign a truth value to** $\boldsymbol{\varphi_i}$ (and thus to some of its atoms)

- instead of *one* $\mu^{\mathbf{A}}$, the solver enumerates *many* $\mu^{\mathbf{A}'}$ that are minimal for $\exists \mathbf{B}.\mathsf{CNF}_{\mathsf{Ts}}(\varphi)$ but not for φ

The impact of Tseitin encoding

The problem

$$\mu^{\mathbf{A}} \models \varphi(\mathbf{A}) \not\Rightarrow \mu^{\mathbf{A}} \models \exists \mathbf{B}.\mathsf{CNF}_{\mathsf{Ts}}(\varphi)$$

i.e., it **does not** imply that some $\eta^{\mathbf{B}}$ exists s.t. $\mu^{\mathbf{A}} \cup \eta^{\mathbf{B}} \models \mathsf{CNF}_{\mathsf{Ts}}(\varphi)$.

Intuition

Each $(B_i \leftrightarrow \varphi_i)$ **forces** every partial model $\mu^{\mathbf{A}\prime}$ of $\exists \mathbf{B}.\mathsf{CNF}_{\mathsf{Ts}}(\varphi)$ **to assign a truth value to φ_i** (and thus to some of its atoms)

- instead of *one* $\mu^{\mathbf{A}}$, the solver enumerates *many* $\mu^{\mathbf{A}\prime}$ that are minimal for $\exists \mathbf{B}.\mathsf{CNF}_{\mathsf{Ts}}(\varphi)$ but not for φ
- each $\mu^{\mathbf{A}\prime}$ conjoins to $\mu^{\mathbf{A}}$ one of the (up to $2^{|\mathbf{A}|-|\mu^{\mathbf{A}}|}$) partial assignments that evaluate to either $\top$ or $\bot$ all unassigned φ_is

The impact of Tseitin encoding

The problem

$$\mu^{\mathbf{A}} \models \varphi(\mathbf{A}) \;\not\Rightarrow\; \mu^{\mathbf{A}} \models \exists \mathbf{B}.\mathsf{CNF}_{\mathsf{Ts}}(\varphi)$$

i.e., it **does not** imply that some $\eta^{\mathbf{B}}$ exists s.t. $\mu^{\mathbf{A}} \cup \eta^{\mathbf{B}} \models \mathsf{CNF}_{\mathsf{Ts}}(\varphi)$.

Intuition

Each $(B_i \leftrightarrow \varphi_i)$ **forces** every partial model $\mu^{\mathbf{A}\prime}$ of $\exists \mathbf{B}.\mathsf{CNF}_{\mathsf{Ts}}(\varphi)$ **to assign a truth value to** φ_i (and thus to some of its atoms)

- instead of *one* $\mu^{\mathbf{A}}$, the solver enumerates *many* $\mu^{\mathbf{A}\prime}$ that are minimal for $\exists \mathbf{B}.\mathsf{CNF}_{\mathsf{Ts}}(\varphi)$ but not for φ
- each $\mu^{\mathbf{A}\prime}$ conjoins to $\mu^{\mathbf{A}}$ one of the (up to $2^{|\mathbf{A}|-|\mu^{\mathbf{A}}|}$) partial assignments that evaluate to either $\top$ or $\bot$ all unassigned φ_is

$\Longrightarrow$ **the solver is forced to enumerate many more models than necessary!**

$$
\begin{aligned}
\mathsf{CNF}_{\mathsf{Ts}}(\varphi) \stackrel{\text{def}}{=} & (\neg B_1 \vee A_1) \wedge (\neg B_1 \vee A_2) \wedge (B_1 \vee \neg A_1 \vee \neg A_2) & \wedge \quad & //(B_1 \leftrightarrow (A_1 \wedge A_2)) \\
& (B_2 \vee \neg A_3) \wedge (B_2 \vee \neg A_4) \wedge (\neg B_2 \vee A_3 \vee A_4) & \wedge \quad & //(B_2 \leftrightarrow (A_3 \vee A_4)) \\
& (B_3 \vee \neg A_5) \wedge (B_3 \vee \neg A_6) \wedge (\neg B_3 \vee A_5 \vee A_6) & \wedge \quad & //(B_3 \leftrightarrow (A_5 \vee A_6)) \\
& (\neg B_4 \vee B_2) \wedge (\neg B_4 \vee B_3) \wedge (B_4 \vee \neg B_2 \vee \neg B_3) & \wedge \quad & //(B_4 \leftrightarrow (B_2 \wedge B_3)) \\
& (\neg B_5 \vee B_4 \vee \neg A_7) \wedge (\neg B_5 \vee \neg B_4 \vee A_7) & \wedge \quad & //(B_5 \leftrightarrow (B_4 \leftrightarrow A_7)) \\
& (B_5 \vee B_4 \vee A_7) \wedge (B_5 \vee \neg B_4 \vee \neg A_7) & \wedge \quad & \\
& (B_1 \vee B_5) & &
\end{aligned}
$$

Instead of a single $\mu^{\mathbf{A}} \stackrel{\text{def}}{=} \{\neg A_3, \neg A_4, \neg A_7\}$, $\quad \eta^{\mathbf{B}} \stackrel{\text{def}}{=} \{\neg B_2, \neg B_4, B_5, \ldots\}$

$$
\text{Enumerate } 9\ \mu^{\mathbf{A}}\text{'s} \quad
\begin{array}{lcccccccl}
 & \multicolumn{2}{c}{\overbrace{\hphantom{\neg A_1, \quad \neg A_2,}}^{B_1}} & & & \multicolumn{2}{c}{\overbrace{\hphantom{\neg A_5, \quad \neg A_6,}}^{B_3}} & & \\
\{ & \neg A_1, & & \neg A_3, & \neg A_4, & \neg A_5, & \neg A_6, & \neg A_7 & \} \quad //(\{\neg B_1, \neg B_3\}) \\
\{ & \neg A_1, & & \neg A_3, & \neg A_4, & A_5, & & \neg A_7 & \} \quad //(\{\neg B_1, B_3\}) \\
\{ & \neg A_1, & & \neg A_3, & \neg A_4, & \neg A_5, & A_6, & \neg A_7 & \} \quad //(\{\neg B_1, B_3\}) \\
\{ & A_1, & \neg A_2, & \neg A_3, & \neg A_4, & \neg A_5, & \neg A_6, & \neg A_7 & \} \quad //(\{\neg B_1, \neg B_3\}) \\
\{ & A_1, & \neg A_2, & \neg A_3, & \neg A_4, & A_5, & & \neg A_7 & \} \quad //(\{\neg B_1, B_3\}) \\
\{ & A_1, & \neg A_2, & \neg A_3, & \neg A_4, & \neg A_5, & A_6, & \neg A_7 & \} \quad //(\{\neg B_1, B_3\}) \\
\{ & A_1, & A_2, & \neg A_3, & \neg A_4, & \neg A_5, & \neg A_6, & \neg A_7 & \} \quad //(\{B_1, \neg B_3\}) \\
\{ & A_1, & A_2, & \neg A_3, & \neg A_4, & A_5, & & \neg A_7 & \} \quad //(\{B_1, B_3\}) \\
\{ & A_1, & A_2, & \neg A_3, & \neg A_4, & \neg A_5, & A_6, & \neg A_7 & \} \quad //(\{B_1, B_3\})
\end{array}
$$

Plaisted and Greenbaum encoding

Consider:

- a Propositional formula $\varphi(\mathbf{A})$
- its P&G encoding $\mathsf{CNF}_{\mathsf{PG}}(\varphi)(\mathbf{A} \cup \mathbf{B})$ where $\mathbf{B}$ is the set of *fresh* labels
- **Intuition**: $\mathsf{CNF}_{\mathsf{PG}}(\varphi)$ recursively rewritten as $\mathsf{CNF}_{\mathsf{PG}}(\varphi[\varphi_i|B_i]) \wedge \mathsf{CNF}_{\mathsf{PG}}(B_i \bowtie \varphi_i)$, φ_i subformula, B_i fresh, $\bowtie \in \{\rightarrow, \leftarrow, \leftrightarrow\}$ iff $pol(\varphi_i) \in \{\text{positive}, \text{negative}, \text{both}\}$ resp.

Plaisted and Greenbaum encoding

Consider:

- a Propositional formula $\varphi(\mathbf{A})$
- its P&G encoding $\mathsf{CNF}_{\mathsf{PG}}(\varphi)(\mathbf{A} \cup \mathbf{B})$ where $\mathbf{B}$ is the set of *fresh* labels
- **Intuition**: $\mathsf{CNF}_{\mathsf{PG}}(\varphi)$ recursively rewritten as $\mathsf{CNF}_{\mathsf{PG}}(\varphi[\varphi_i | B_i]) \wedge \mathsf{CNF}_{\mathsf{PG}}(B_i \bowtie \varphi_i)$, φ_i subformula, B_i fresh, $\bowtie \in \{\rightarrow, \leftarrow, \leftrightarrow\}$ iff $pol(\varphi_i) \in \{\text{positive}, \text{negative}, \text{both}\}$ resp.

Then

- $\varphi(\mathbf{A}) \equiv \exists \mathbf{B}.\mathsf{CNF}_{\mathsf{PG}}(\varphi)(\mathbf{A} \cup \mathbf{B})$
- $\mathcal{TA}(\varphi(\mathbf{A}))$ can be enumerated as $\mathcal{TA}(\exists \mathbf{B}.\mathsf{CNF}_{\mathsf{PG}}(\varphi)(\mathbf{A} \cup \mathbf{B}))$
  $\Longrightarrow$ project the models of $\mathsf{CNF}_{\mathsf{PG}}(\varphi)$ onto $\mathbf{A}$ only

Let $\varphi \stackrel{\text{def}}{=} \underbrace{(A_1 \land A_2)}_{B_1} \lor \underbrace{(\underbrace{(\underbrace{(A_3 \lor A_4)}_{B_2} \land \underbrace{(A_5 \lor A_6)}_{B_3})}_{B_4} \leftrightarrow A_7)}_{B_5}$

$$
\begin{aligned}
\mathsf{CNF_{PG}}(\varphi) \stackrel{\text{def}}{=} & (\neg B_1 \lor A_1) \land (\neg B_1 \lor A_2) && \land \quad //(B_1 \rightarrow (A_1 \land A_2)) \\
& (B_2 \lor \neg A_3) \land (B_2 \lor \neg A_4) \land (\neg B_2 \lor A_3 \lor A_4) && \land \quad //(B_2 \leftrightarrow (A_3 \lor A_4)) \\
& (B_3 \lor \neg A_5) \land (B_3 \lor \neg A_6) \land (\neg B_3 \lor A_5 \lor A_6) && \land \quad //(B_3 \leftrightarrow (A_5 \lor A_6)) \\
& (\neg B_4 \lor B_2) \land (\neg B_4 \lor B_3) \land (B_4 \lor \neg B_2 \lor \neg B_3) && \land \quad //(B_4 \leftrightarrow (B_2 \land B_3)) \\
& (\neg B_5 \lor B_4 \lor \neg A_7) \land (\neg B_5 \lor \neg B_4 \lor A_7) && \land \quad //(B_5 \leftrightarrow (B_4 \leftrightarrow A_7)) \\
& (B_5 \lor B_4 \lor A_7) \land (B_5 \lor \neg B_4 \lor \neg A_7) && \land \\
& (B_1 \lor B_5)
\end{aligned}
$$

Let $\varphi \stackrel{\text{def}}{=} \underbrace{\overbrace{(A_1 \wedge A_2)}^{B_1}} \vee \overbrace{(\overbrace{(\overbrace{(A_3 \vee A_4)}^{B_2} \wedge \overbrace{(A_5 \vee A_6)}^{B_3})}^{B_4} \leftrightarrow A_7)}^{B_5}$

$$
\begin{array}{llr}
\mathsf{CNF_{PG}}(\varphi) \stackrel{\text{def}}{=} & (\neg B_1 \vee A_1) \wedge (\neg B_1 \vee A_2) & \wedge \quad //(B_1 \rightarrow (A_1 \wedge A_2)) \\
& (B_2 \vee \neg A_3) \wedge (B_2 \vee \neg A_4) \wedge (\neg B_2 \vee A_3 \vee A_4) & \wedge \quad //(B_2 \leftrightarrow (A_3 \vee A_4)) \\
& (B_3 \vee \neg A_5) \wedge (B_3 \vee \neg A_6) \wedge (\neg B_3 \vee A_5 \vee A_6) & \wedge \quad //(B_3 \leftrightarrow (A_5 \vee A_6)) \\
& (\neg B_4 \vee B_2) \wedge (\neg B_4 \vee B_3) \wedge (B_4 \vee \neg B_2 \vee \neg B_3) & \wedge \quad //(B_4 \leftrightarrow (B_2 \wedge B_3)) \\
& (\neg B_5 \vee B_4 \vee \neg A_7) \wedge (\neg B_5 \vee \neg B_4 \vee A_7) & \wedge \quad //(B_5 \leftrightarrow (B_4 \leftrightarrow A_7)) \\
& (B_5 \vee B_4 \vee A_7) \wedge (B_5 \vee \neg B_4 \vee \neg A_7) & \wedge \\
& (B_1 \vee B_5) &
\end{array}
$$

Consider again $\mu^{\mathbf{A}} \stackrel{\text{def}}{=} \{\neg A_3, \neg A_4, \neg A_7\}$ s.t. $\mu^{\mathbf{A}} \models \varphi$. Does $\mu^{\mathbf{A}} \models \exists \mathbf{B}.\mathsf{CNF_{PG}}(\varphi)$?

Let $\varphi \stackrel{\text{def}}{=} \overbrace{(A_1 \wedge A_2)}^{B_1} \vee \overbrace{(\overbrace{(\overbrace{(A_3 \vee A_4)}^{B_2} \wedge \overbrace{(A_5 \vee A_6)}^{B_3})}^{B_4} \leftrightarrow A_7)}^{B_5}$

$$
\begin{array}{lll}
\mathsf{CNF_{PG}}(\varphi) \stackrel{\text{def}}{=} & (\neg B_1 \vee A_1) \wedge (\neg B_1 \vee A_2) & \wedge \quad //(B_1 \rightarrow (A_1 \wedge A_2)) \\
& (B_2 \vee \neg A_3) \wedge (B_2 \vee \neg A_4) \wedge (\neg B_2 \vee A_3 \vee A_4) & \wedge \quad //(B_2 \leftrightarrow (A_3 \vee A_4)) \\
& (B_3 \vee \neg A_5) \wedge (B_3 \vee \neg A_6) \wedge (\neg B_3 \vee A_5 \vee A_6) & \wedge \quad //(B_3 \leftrightarrow (A_5 \vee A_6)) \\
& (\neg B_4 \vee B_2) \wedge (\neg B_4 \vee B_3) \wedge (B_4 \vee \neg B_2 \vee \neg B_3) & \wedge \quad //(B_4 \leftrightarrow (B_2 \wedge B_3)) \\
& (\neg B_5 \vee B_4 \vee \neg A_7) \wedge (\neg B_5 \vee \neg B_4 \vee A_7) & \wedge \quad //(B_5 \leftrightarrow (B_4 \leftrightarrow A_7)) \\
& (B_5 \vee B_4 \vee A_7) \wedge (B_5 \vee \neg B_4 \vee \neg A_7) & \wedge \\
& (B_1 \vee B_5) &
\end{array}
$$

Consider again $\mu^{\mathbf{A}} \stackrel{\text{def}}{=} \{\neg A_3, \neg A_4, \neg A_7\}$ s.t. $\mu^{\mathbf{A}} \models \varphi$. Does $\mu^{\mathbf{A}} \models \exists \mathbf{B}.\mathsf{CNF_{PG}}(\varphi)$?

E.g., $\eta^{\mathbf{B}} \stackrel{\text{def}}{=} \{\neg B_2, \ldots\}$

Let $\varphi \stackrel{\text{def}}{=} \underbrace{(A_1 \wedge A_2)}_{B_1} \vee \underbrace{(\underbrace{(\underbrace{(A_3 \vee A_4)}_{B_2} \wedge \underbrace{(A_5 \vee A_6)}_{B_3})}_{B_4} \leftrightarrow A_7)}_{B_5}$

$$
\begin{array}{lll}
\mathsf{CNF}_{\mathsf{PG}}(\varphi) \stackrel{\text{def}}{=} & (\neg B_1 \vee A_1) \wedge (\neg B_1 \vee A_2) & \wedge \quad //(B_1 \rightarrow (A_1 \wedge A_2)) \\
& (B_2 \vee \neg A_3) \wedge (B_2 \vee \neg A_4) \wedge (\neg B_2 \vee A_3 \vee A_4) & \wedge \quad //(B_2 \leftrightarrow (A_3 \vee A_4)) \\
& (B_3 \vee \neg A_5) \wedge (B_3 \vee \neg A_6) \wedge (\neg B_3 \vee A_5 \vee A_6) & \wedge \quad //(B_3 \leftrightarrow (A_5 \vee A_6)) \\
& (\neg B_4 \vee B_2) \wedge (\neg B_4 \vee B_3) \wedge (B_4 \vee \neg B_2 \vee \neg B_3) & \wedge \quad //(B_4 \leftrightarrow (B_2 \wedge B_3)) \\
& (\neg B_5 \vee B_4 \vee \neg A_7) \wedge (\neg B_5 \vee \neg B_4 \vee A_7) & \wedge \quad //(B_5 \leftrightarrow (B_4 \leftrightarrow A_7)) \\
& (B_5 \vee B_4 \vee A_7) \wedge (B_5 \vee \neg B_4 \vee \neg A_7) & \wedge \\
& (B_1 \vee B_5) &
\end{array}
$$

Consider again $\mu^{\mathbf{A}} \stackrel{\text{def}}{=} \{\neg A_3, \neg A_4, \neg A_7\}$ s.t. $\mu^{\mathbf{A}} \models \varphi$. Does $\mu^{\mathbf{A}} \models \exists \mathbf{B}.\mathsf{CNF}_{\mathsf{PG}}(\varphi)$?

E.g., $\eta^{\mathbf{B}} \stackrel{\text{def}}{=} \{\neg B_2, \neg B_4, \ldots\}$

Let $\varphi \stackrel{\text{def}}{=} \underbrace{(A_1 \wedge A_2)}_{B_1} \vee (\underbrace{(\underbrace{(A_3 \vee A_4)}_{B_2} \wedge \underbrace{(A_5 \vee A_6)}_{B_3})}_{B_4} \leftrightarrow A_7)$ (with B_5 labelling $((A_3 \vee A_4) \wedge (A_5 \vee A_6)) \leftrightarrow A_7$)

$$
\begin{array}{llr}
\mathsf{CNF}_{\mathsf{PG}}(\varphi) \stackrel{\text{def}}{=} & (\neg B_1 \vee A_1) \wedge (\neg B_1 \vee A_2) & \wedge \quad //(B_1 \rightarrow (A_1 \wedge A_2)) \\
& (B_2 \vee \neg A_3) \wedge (B_2 \vee \neg A_4) \wedge (\neg B_2 \vee A_3 \vee A_4) & \wedge \quad //(B_2 \leftrightarrow (A_3 \vee A_4)) \\
& (B_3 \vee \neg A_5) \wedge (B_3 \vee \neg A_6) \wedge (\neg B_3 \vee A_5 \vee A_6) & \wedge \quad //(B_3 \leftrightarrow (A_5 \vee A_6)) \\
& (\neg B_4 \vee B_2) \wedge (\neg B_4 \vee B_3) \wedge (B_4 \vee \neg B_2 \vee \neg B_3) & \wedge \quad //(B_4 \leftrightarrow (B_2 \wedge B_3)) \\
& (\neg B_5 \vee B_4 \vee \neg A_7) \wedge (\neg B_5 \vee \neg B_4 \vee A_7) & \wedge \quad //(B_5 \leftrightarrow (B_4 \leftrightarrow A_7)) \\
& (B_5 \vee B_4 \vee A_7) \wedge (B_5 \vee \neg B_4 \vee \neg A_7) & \wedge \\
& (B_1 \vee B_5) &
\end{array}
$$

Consider again $\mu^{\mathbf{A}} \stackrel{\text{def}}{=} \{\neg A_3, \neg A_4, \neg A_7\}$ s.t. $\mu^{\mathbf{A}} \models \varphi$. Does $\mu^{\mathbf{A}} \models \exists \mathbf{B}.\mathsf{CNF}_{\mathsf{PG}}(\varphi)$?

E.g., $\eta^{\mathbf{B}} \stackrel{\text{def}}{=} \{\neg B_2, \neg B_4, B_5, \ldots\}$

$$\text{Let } \varphi \stackrel{\text{def}}{=} \underbrace{\boxed{(A_1 \wedge A_2)}}_{B_1} \vee \underbrace{\Big(\underbrace{\big(\underbrace{\boxed{(A_3 \vee A_4)}}_{B_2} \wedge \underbrace{\boxed{(A_5 \vee A_6)}}_{B_3} \big)}_{B_4} \leftrightarrow A_7 \Big)}_{B_5}$$

$$\begin{array}{rll}
\mathsf{CNF_{PG}}(\varphi) \stackrel{\text{def}}{=} & (\neg B_1 \vee A_1) \wedge (\neg B_1 \vee A_2) & \wedge \quad //(B_1 \rightarrow (A_1 \wedge A_2)) \\
& (B_2 \vee \neg A_3) \wedge (B_2 \vee \neg A_4) \wedge (\neg B_2 \vee A_3 \vee A_4) & \wedge \quad //(B_2 \leftrightarrow (A_3 \vee A_4)) \\
& (B_3 \vee \neg A_5) \wedge (B_3 \vee \neg A_6) \wedge (\neg B_3 \vee A_5 \vee A_6) & \wedge \quad //(B_3 \leftrightarrow (A_5 \vee A_6)) \\
& (\neg B_4 \vee B_2) \wedge (\neg B_4 \vee B_3) \wedge (B_4 \vee \neg B_2 \vee \neg B_3) & \wedge \quad //(B_4 \leftrightarrow (B_2 \wedge B_3)) \\
& (\neg B_5 \vee B_4 \vee \neg A_7) \wedge (\neg B_5 \vee \neg B_4 \vee A_7) & \wedge \quad //(B_5 \leftrightarrow (B_4 \leftrightarrow A_7)) \\
& (B_5 \vee B_4 \vee A_7) \wedge (B_5 \vee \neg B_4 \vee \neg A_7) & \wedge \\
& (B_1 \vee B_5) &
\end{array}$$

Consider again $\mu^{\mathbf{A}} \stackrel{\text{def}}{=} \{\neg A_3, \neg A_4, \neg A_7\}$ s.t. $\mu^{\mathbf{A}} \models \varphi$. Does $\mu^{\mathbf{A}} \models \exists \mathbf{B}.\mathsf{CNF_{PG}}(\varphi)$?

E.g., $\eta^{\mathbf{B}} \stackrel{\text{def}}{=} \{\neg B_2, \neg B_4, B_5, \neg B_1, \ldots\}$

- **By assigning $\eta^{\mathbf{B}}(B_1) = \bot$ we can avoid assigning a truth value to A_1 and A_2!**

Let $\varphi \stackrel{\text{def}}{=} \underbrace{(A_1 \wedge A_2)}_{B_1} \vee \underbrace{(\underbrace{(\underbrace{(A_3 \vee A_4)}_{B_2} \wedge \underbrace{(A_5 \vee A_6)}_{B_3})}_{B_4} \leftrightarrow A_7)}_{B_5}$

$$
\begin{array}{llll}
\mathsf{CNF}_{\mathsf{PG}}(\varphi) \stackrel{\text{def}}{=} & (\neg B_1 \vee A_1) \wedge (\neg B_1 \vee A_2) & \wedge & //(B_1 \rightarrow (A_1 \wedge A_2)) \\
& (B_2 \vee \neg A_3) \wedge (B_2 \vee \neg A_4) \wedge (\neg B_2 \vee A_3 \vee A_4) & \wedge & //(B_2 \leftrightarrow (A_3 \vee A_4)) \\
& (B_3 \vee \neg A_5) \wedge (B_3 \vee \neg A_6) \wedge (\neg B_3 \vee A_5 \vee A_6) & \wedge & //(B_3 \leftrightarrow (A_5 \vee A_6)) \\
& (\neg B_4 \vee B_2) \wedge (\neg B_4 \vee B_3) \wedge (B_4 \vee \neg B_2 \vee \neg B_3) & \wedge & //(B_4 \leftrightarrow (B_2 \wedge B_3)) \\
& (\neg B_5 \vee B_4 \vee \neg A_7) \wedge (\neg B_5 \vee \neg B_4 \vee A_7) & \wedge & //(B_5 \leftrightarrow (B_4 \leftrightarrow A_7)) \\
& (B_5 \vee B_4 \vee A_7) \wedge (B_5 \vee \neg B_4 \vee \neg A_7) & \wedge & \\
& (B_1 \vee B_5) & &
\end{array}
$$

Consider again $\mu^{\mathbf{A}} \stackrel{\text{def}}{=} \{\neg A_3, \neg A_4, \neg A_7\}$ s.t. $\mu^{\mathbf{A}} \models \varphi$. Does $\mu^{\mathbf{A}} \models \exists \mathbf{B}.\mathsf{CNF}_{\mathsf{PG}}(\varphi)$?

E.g., $\eta^{\mathbf{B}} \stackrel{\text{def}}{=} \{\neg B_2, \neg B_4, B_5, \neg B_1, \neg B_3\}$

- **By assigning $\eta^{\mathbf{B}}(B_1) = \bot$ we can avoid assigning a truth value to A_1 and A_2!**
- However, still $\mu^{\mathbf{A}} \not\models \exists \mathbf{B}.\mathsf{CNF}_{\mathsf{PG}}(\varphi)$!
  - $(\neg A_5 \wedge \neg A_6)$ is not satisfied

The impact of Plaisted and Greenbaum encoding

Positive side

If φ_i occurs only positively, the clauses representing $(B_i \to \varphi_i)$ can be satisfied by $\eta^{\mathbf{B}}(B_i) = \bot$. [dual for negative occurrences]

The impact of Plaisted and Greenbaum encoding

Positive side

If φ_i occurs only positively, the clauses representing $(B_i \rightarrow \varphi_i)$ can be satisfied by $\eta^{\mathbf{B}}(B_i) = \bot$. [dual for negative occurrences]

Negative side

If φ_i has double polarity $\Longrightarrow$ same problem as with $\mathsf{CNF}_{\mathsf{Ts}}$:
each $\mu^{\mathbf{A}\prime}$ s.t. $\mu^{\mathbf{A}\prime} \models \exists \mathbf{B}.\mathsf{CNF}_{\mathsf{PG}}(\varphi)$ must assign atoms that evaluate φ_i to either $\top$ or $\bot$ because of $(B_i \leftrightarrow \varphi_i)$.

Outline

Combining NNF and Plaisted and Greenbaum encoding

Proposed solution

1. Transform first the input formula into an NNF DAG (linear in size).
2. Apply Plaisted and Greenbaum trasformation to the NNF DAG.

Combining NNF and Plaisted and Greenbaum encoding

Proposed solution

1. Transform first the input formula into an NNF DAG (linear in size).
2. Apply Plaisted and Greenbaum trasformation to the NNF DAG.

- every φ_i with double polarity is converted into **two syntactically-different sub-formulas**, each **occurring only positively**:

$$\varphi_i^+ \stackrel{\text{def}}{=} \mathsf{NNF}(\varphi_i) \text{ and } \varphi_i^- \stackrel{\text{def}}{=} \mathsf{NNF}(\neg\varphi_i)$$

Combining NNF and Plaisted and Greenbaum encoding

Proposed solution

1. Transform first the input formula into an NNF DAG (linear in size).
2. Apply Plaisted and Greenbaum trasformation to the NNF DAG.

- every φ_i with double polarity is converted into **two syntactically-different sub-formulas**, each **occurring only positively**:

$$\varphi_i^+ \stackrel{\text{def}}{=} \mathsf{NNF}(\varphi_i) \text{ and } \varphi_i^- \stackrel{\text{def}}{=} \mathsf{NNF}(\neg\varphi_i)$$

- the two sub-formulas are labelled with two different atoms B_i^+, B_i^- as

$$(B_i^+ \rightarrow \varphi_i^+) \wedge (B_i^- \rightarrow \varphi_i^-)$$

Combining NNF and Plaisted and Greenbaum encoding

Proposed solution

1. Transform first the input formula into an NNF DAG (linear in size).
2. Apply Plaisted and Greenbaum trasformation to the NNF DAG.

- every φ_i with double polarity is converted into **two syntactically-different sub-formulas**, each **occurring only positively**:

$$\varphi_i^+ \stackrel{\text{def}}{=} \mathsf{NNF}(\varphi_i) \text{ and } \varphi_i^- \stackrel{\text{def}}{=} \mathsf{NNF}(\neg\varphi_i)$$

- the two sub-formulas are labelled with two different atoms B_i^+, B_i^- as

$$(B_i^+ \to \varphi_i^+) \wedge (B_i^- \to \varphi_i^-)$$

- also add mutex condition $(\neg B_i^+ \vee \neg B_i^-)$ (not necessary but helps pruning the search)

Let $\varphi \stackrel{\text{def}}{=} (A_1 \wedge A_2) \vee (((A_3 \vee A_4) \wedge (A_5 \vee A_6)) \leftrightarrow A_7)$

Let $\varphi \stackrel{\text{def}}{=} (A_1 \wedge A_2) \vee (((A_3 \vee A_4) \wedge (A_5 \vee A_6)) \leftrightarrow A_7)$

$$\mathsf{NNF}(\varphi) \stackrel{\text{def}}{=} \underbrace{(A_1 \wedge A_2)}_{B_1} \vee \underbrace{(\underbrace{(\underbrace{(\underbrace{(\neg A_3 \wedge \neg A_4)}_{B_2^-} \vee \underbrace{(\neg A_5 \wedge \neg A_6)}_{B_3^-})}_{B_4^-} \vee A_7)}_{B_5} \wedge \underbrace{(\underbrace{(\underbrace{(A_3 \vee A_4)}_{B_2^+} \wedge \underbrace{(A_5 \vee A_6)}_{B_3^+})}_{B_4^+} \vee \neg A_7)}_{B_6})}_{B_7}$$

$$\text{Let } \varphi \stackrel{\text{def}}{=} (A_1 \wedge A_2) \vee (((A_3 \vee A_4) \wedge (A_5 \vee A_6)) \leftrightarrow A_7)$$

$$\mathsf{NNF}(\varphi) \stackrel{\text{def}}{=} \overset{B_1}{(A_1 \wedge A_2)} \vee \overset{B_7}{\left(\overset{B_5}{\left(\overset{B_4^-}{\left(\overset{B_2^-}{(\neg A_3 \wedge \neg A_4)} \vee \overset{B_3^-}{(\neg A_5 \wedge \neg A_6)} \right)} \vee A_7 \right)} \wedge \overset{B_6}{\left(\overset{B_4^+}{\left(\overset{B_2^+}{(A_3 \vee A_4)} \wedge \overset{B_3^+}{(A_5 \vee A_6)} \right)} \vee \neg A_7 \right)} \right)}$$

$$\begin{aligned}
\mathsf{CNF}_{\mathsf{PG}}(\mathsf{NNF}(\varphi)) \stackrel{\text{def}}{=} & (\neg B_1 \vee A_1) \wedge (\neg B_1 \vee A_2) \wedge && //(B_1 \rightarrow (A_1 \wedge A_2)) \\
& (\neg B_2^- \vee \neg A_3) \wedge (\neg B_2^- \vee \neg A_4) \wedge && //(B_2^- \rightarrow (\neg A_3 \wedge \neg A_4)) \\
& (\neg B_3^- \vee \neg A_5) \wedge (\neg B_3^- \vee \neg A_6) \wedge && //(B_3^- \rightarrow (\neg A_5 \wedge \neg A_6)) \\
& (\neg B_4^- \vee B_2^- \vee B_3^-) \wedge && //(B_4^- \rightarrow (B_2^- \vee B_3^-)) \\
& (\neg B_5 \vee B_4^- \vee A_7) \wedge && //(B_5 \rightarrow (B_4^- \vee A_7)) \\
& (\neg B_2^+ \vee A_3 \vee A_4) \wedge && //(B_2^+ \rightarrow (A_3 \vee A_4)) \\
& (\neg B_3^+ \vee A_5 \vee A_6) \wedge && //(B_3^+ \rightarrow (A_5 \vee A_6)) \\
& (\neg B_4^+ \vee B_2^+) \wedge (\neg B_4^+ \vee B_3^+) \wedge && //(B_4^+ \rightarrow (B_2^+ \wedge B_3^+)) \\
& (\neg B_6 \vee B_4^+ \vee \neg A_7) \wedge && //(B_6 \rightarrow (B_4^+ \vee \neg A_7)) \\
& (\neg B_7 \vee B_5) \wedge (\neg B_7 \vee B_6) \wedge && //(B_7 \rightarrow (B_5 \wedge B_6)) \\
& (B_1 \vee B_7) \wedge && \\
& (\neg B_2^+ \vee \neg B_2^-) \wedge (\neg B_3^+ \vee \neg B_3^-) \wedge (\neg B_4^+ \vee \neg B_4^-) &&
\end{aligned}$$

Let $\varphi \stackrel{\text{def}}{=} (A_1 \wedge A_2) \vee (((A_3 \vee A_4) \wedge (A_5 \vee A_6)) \leftrightarrow A_7)$

$$\mathsf{NNF}(\varphi) \stackrel{\text{def}}{=} \overset{B_1}{(A_1 \wedge A_2)} \vee \overset{B_7}{\left(\overset{B_5}{\left(\overset{B_4^-}{\left(\overset{B_2^-}{(\neg A_3 \wedge \neg A_4)} \vee \overset{B_3^-}{(\neg A_5 \wedge \neg A_6)} \right)} \vee A_7 \right)} \wedge \overset{B_6}{\left(\overset{B_4^+}{\left(\overset{B_2^+}{(A_3 \vee A_4)} \wedge \overset{B_3^+}{(A_5 \vee A_6)} \right)} \vee \neg A_7 \right)} \right)}$$

$$\begin{array}{lll}
\mathsf{CNF_{PG}}(\mathsf{NNF}(\varphi)) \stackrel{\text{def}}{=} & (\neg B_1 \vee A_1) \wedge (\neg B_1 \vee A_2) \wedge & //(B_1 \to (A_1 \wedge A_2)) \\
& (\neg B_2^- \vee \neg A_3) \wedge (\neg B_2^- \vee \neg A_4) \wedge & //(B_2^- \to (\neg A_3 \wedge \neg A_4)) \\
& (\neg B_3^- \vee \neg A_5) \wedge (\neg B_3^- \vee \neg A_6) \wedge & //(B_3^- \to (\neg A_5 \wedge \neg A_6)) \\
& (\neg B_4^- \vee B_2^- \vee B_3^-) \wedge & //(B_4^- \to (B_2^- \vee B_3^-)) \\
& (\neg B_5 \vee B_4^- \vee A_7) \wedge & //(B_5 \to (B_4^- \vee A_7)) \\
& (\neg B_2^+ \vee A_3 \vee A_4) \wedge & //(B_2^+ \to (A_3 \vee A_4)) \\
& (\neg B_3^+ \vee A_5 \vee A_6) \wedge & //(B_3^+ \to (A_5 \vee A_6)) \\
& (\neg B_4^+ \vee B_2^+) \wedge (\neg B_4^+ \vee B_3^+) \wedge & //(B_4^+ \to (B_2^+ \wedge B_3^+)) \\
& (\neg B_6 \vee B_4^+ \vee \neg A_7) \wedge & //(B_6 \to (B_4^+ \vee \neg A_7)) \\
& (\neg B_7 \vee B_5) \wedge (\neg B_7 \vee B_6) \wedge & //(B_7 \to (B_5 \wedge B_6)) \\
& (B_1 \vee B_7) \wedge & \\
& (\neg B_2^+ \vee \neg B_2^-) \wedge (\neg B_3^+ \vee \neg B_3^-) \wedge (\neg B_4^+ \vee \neg B_4^-) &
\end{array}$$

Let $\mu^{\mathbf{A}} \stackrel{\text{def}}{=} \{\neg A_3, \neg A_4, \neg A_7\}$
$\mu^{\mathbf{A}} \models \varphi$
$\mu^{\mathbf{A}} \models \mathsf{NNF}(\varphi)$
$\mu^{\mathbf{A}} \models \exists \mathbf{B}.\mathsf{CNF_{PG}}(\mathsf{NNF}(\varphi))$

$$\text{Let } \varphi \stackrel{\text{def}}{=} (A_1 \wedge A_2) \vee (((A_3 \vee A_4) \wedge (A_5 \vee A_6)) \leftrightarrow A_7)$$

$$\mathsf{NNF}(\varphi) \stackrel{\text{def}}{=} \overbrace{(A_1 \wedge A_2)}^{B_1} \vee \overbrace{(\overbrace{(\overbrace{(\overbrace{(\neg A_3 \wedge \neg A_4)}^{B_2^-} \vee \overbrace{(\neg A_5 \wedge \neg A_6)}^{B_3^-})}^{B_4^-} \vee A_7)}^{B_5} \wedge \overbrace{(\overbrace{(\overbrace{(A_3 \vee A_4)}^{B_2^+} \wedge \overbrace{(A_5 \vee A_6)}^{B_3^+})}^{B_4^+} \vee \neg A_7)}^{B_6})}^{B_7}$$

$$\begin{array}{lll}
\mathsf{CNF_{PG}}(\mathsf{NNF}(\varphi)) \stackrel{\text{def}}{=} & (\neg B_1 \vee A_1) \wedge (\neg B_1 \vee A_2) \wedge & //(B_1 \rightarrow (A_1 \wedge A_2)) \\
& (\neg B_2^- \vee \neg A_3) \wedge (\neg B_2^- \vee \neg A_4) \wedge & //(B_2^- \rightarrow (\neg A_3 \wedge \neg A_4)) \\
& (\neg B_3^- \vee \neg A_5) \wedge (\neg B_3^- \vee \neg A_6) \wedge & //(B_3^- \rightarrow (\neg A_5 \wedge \neg A_6)) \\
& (\neg B_4^- \vee B_2^- \vee B_3^-) \wedge & //(B_4^- \rightarrow (B_2^- \vee B_3^-)) \\
& (\neg B_5 \vee B_4^- \vee A_7) \wedge & //(B_5 \rightarrow (B_4^- \vee A_7)) \\
& (\neg B_2^+ \vee A_3 \vee A_4) \wedge & //(B_2^+ \rightarrow (A_3 \vee A_4)) \\
& (\neg B_3^+ \vee A_5 \vee A_6) \wedge & //(B_3^+ \rightarrow (A_5 \vee A_6)) \\
& (\neg B_4^+ \vee B_2^+) \wedge (\neg B_4^+ \vee B_3^+) \wedge & //(B_4^+ \rightarrow (B_2^+ \wedge B_3^+)) \\
& (\neg B_6 \vee B_4^+ \vee \neg A_7) \wedge & //(B_6 \rightarrow (B_4^+ \vee \neg A_7)) \\
& (\neg B_7 \vee B_5) \wedge (\neg B_7 \vee B_6) \wedge & //(B_7 \rightarrow (B_5 \wedge B_6)) \\
& (B_1 \vee B_7) \wedge & \\
& (\neg B_2^+ \vee \neg B_2^-) \wedge (\neg B_3^+ \vee \neg B_3^-) \wedge (\neg B_4^+ \vee \neg B_4^-) &
\end{array}$$

Let $\mu^{\mathbf{A}} \stackrel{\text{def}}{=} \{\neg A_3, \neg A_4, \neg A_7\}$
$\mu^{\mathbf{A}} \models \varphi$
$\mu^{\mathbf{A}} \models \mathsf{NNF}(\varphi)$
$\mu^{\mathbf{A}} \models \exists \mathbf{B}.\mathsf{CNF_{PG}}(\mathsf{NNF}(\varphi))$
E.g.

$\eta^{\mathbf{B}} \stackrel{\text{def}}{=} \{ B_7, B_5, B_6, B_2^-, B_4^-, \ldots\}$

Let $\varphi \stackrel{\text{def}}{=} (A_1 \wedge A_2) \vee (((A_3 \vee A_4) \wedge (A_5 \vee A_6)) \leftrightarrow A_7)$

$$\mathsf{NNF}(\varphi) \stackrel{\text{def}}{=} \overset{B_1}{(A_1 \wedge A_2)} \vee \overbrace{(\overbrace{(\overbrace{(\overset{B_2^-}{(\neg A_3 \wedge \neg A_4)} \vee \overset{B_3^-}{(\neg A_5 \wedge \neg A_6)})}^{B_4^-} \vee A_7)}^{B_5} \wedge \overbrace{(\overbrace{(\overset{B_2^+}{(A_3 \vee A_4)} \wedge \overset{B_3^+}{(A_5 \vee A_6)})}^{B_4^+} \vee \neg A_7)}^{B_6})}^{B_7}$$

$$\begin{array}{lll}
\mathsf{CNF}_{\mathsf{PG}}(\mathsf{NNF}(\varphi)) \stackrel{\text{def}}{=} & (\neg B_1 \vee A_1) \wedge (\neg B_1 \vee A_2) \wedge & //(B_1 \rightarrow (A_1 \wedge A_2)) \\
& (\neg B_2^- \vee \neg A_3) \wedge (\neg B_2^- \vee \neg A_4) \wedge & //(B_2^- \rightarrow (\neg A_3 \wedge \neg A_4)) \\
& (\neg B_3^- \vee \neg A_5) \wedge (\neg B_3^- \vee \neg A_6) \wedge & //(B_3^- \rightarrow (\neg A_5 \wedge \neg A_6)) \\
& (\neg B_4^- \vee B_2^- \vee B_3^-) \wedge & //(B_4^- \rightarrow (B_2^- \vee B_3^-)) \\
& (\neg B_5 \vee B_4^- \vee A_7) \wedge & //(B_5 \rightarrow (B_4^- \vee A_7)) \\
& (\neg B_2^+ \vee A_3 \vee A_4) \wedge & //(B_2^+ \rightarrow (A_3 \vee A_4)) \\
& (\neg B_3^+ \vee A_5 \vee A_6) \wedge & //(B_3^+ \rightarrow (A_5 \vee A_6)) \\
& (\neg B_4^+ \vee B_2^+) \wedge (\neg B_4^+ \vee B_3^+) \wedge & //(B_4^+ \rightarrow (B_2^+ \wedge B_3^+)) \\
& (\neg B_6 \vee B_4^+ \vee \neg A_7) \wedge & //(B_6 \rightarrow (B_4^+ \vee \neg A_7)) \\
& (\neg B_7 \vee B_5) \wedge (\neg B_7 \vee B_6) \wedge & //(B_7 \rightarrow (B_5 \wedge B_6)) \\
& (B_1 \vee B_7) \wedge & \\
& (\neg B_2^+ \vee \neg B_2^-) \wedge (\neg B_3^+ \vee \neg B_3^-) \wedge (\neg B_4^+ \vee \neg B_4^-) &
\end{array}$$

Let $\mu^{\mathbf{A}} \stackrel{\text{def}}{=} \{\neg A_3, \neg A_4, \neg A_7\}$

$\mu^{\mathbf{A}} \models \varphi$

$\mu^{\mathbf{A}} \models \mathsf{NNF}(\varphi)$

$\mu^{\mathbf{A}} \models \exists \mathbf{B}.\mathsf{CNF}_{\mathsf{PG}}(\mathsf{NNF}(\varphi))$

E.g.

$$\eta^{\mathbf{B}} \stackrel{\text{def}}{=} \{ B_7, B_5, B_6, B_2^-, B_4^-, \neg B_1, \neg B_3^-, \neg B_3^+, \neg B_2^+, \neg B_4^+ \}$$

Combining NNF and Plaisted and Greenbaum CNF

$\mathsf{NNF} + \mathsf{CNF}_{\mathsf{PG}}$ allows for assigning "**don't care**" value also to φ_is with **double polarity**:

- $\varphi_i^+ \stackrel{\text{def}}{=} \mathsf{NNF}(\varphi_i)$ and $\varphi_i^- \stackrel{\text{def}}{=} \mathsf{NNF}(\neg\varphi_i)$ occur only positively in $\mathsf{NNF}(\varphi)$

[3] G. Masina, G. Spallitta, and R. Sebastiani (2023-06). *On CNF Conversion for SAT Enumeration.* arXiv: 2303.14971

Combining NNF and Plaisted and Greenbaum CNF

$\mathsf{NNF} + \mathsf{CNF_{PG}}$ allows for assigning "**don't care**" value also to φ_is with **double polarity**:

- $\varphi_i^+ \stackrel{\text{def}}{=} \mathsf{NNF}(\varphi_i)$ and $\varphi_i^- \stackrel{\text{def}}{=} \mathsf{NNF}(\neg\varphi_i)$ occur only positively in $\mathsf{NNF}(\varphi)$
- instead of adding $(B_i \leftrightarrow \varphi_i)$, add $(B_i^+ \rightarrow \varphi_i^+) \wedge (B_i^- \rightarrow \varphi_i^-)$

[3] G. Masina, G. Spallitta, and R. Sebastiani (2023-06). *On CNF Conversion for SAT Enumeration.* arXiv: 2303.14971

Combining NNF and Plaisted and Greenbaum CNF

$\mathsf{NNF} + \mathsf{CNF_{PG}}$ allows for assigning "**don't care**" value also to φ_is with **double polarity**:

- $\varphi_i^+ \stackrel{\text{def}}{=} \mathsf{NNF}(\varphi_i)$ and $\varphi_i^- \stackrel{\text{def}}{=} \mathsf{NNF}(\neg\varphi_i)$ occur only positively in $\mathsf{NNF}(\varphi)$
- instead of adding $(B_i \leftrightarrow \varphi_i)$, add $(B_i^+ \rightarrow \varphi_i^+) \wedge (B_i^- \rightarrow \varphi_i^-)$
- the clauses defining B_i^+, B_i^- can be satisfied by $\eta^{\mathbf{B}}(B_i^+) = \eta^{\mathbf{B}}(B_i^-) = \perp$ without assigning a truth value to φ_i

[3] G. Masina, G. Spallitta, and R. Sebastiani (2023-06). *On CNF Conversion for SAT Enumeration.* arXiv: 2303.14971

Combining NNF and Plaisted and Greenbaum CNF

$\mathsf{NNF} + \mathsf{CNF_{PG}}$ allows for assigning "**don't care**" value also to φ_is with **double polarity**:

- $\varphi_i^+ \stackrel{\text{def}}{=} \mathsf{NNF}(\varphi_i)$ and $\varphi_i^- \stackrel{\text{def}}{=} \mathsf{NNF}(\neg\varphi_i)$ occur only positively in $\mathsf{NNF}(\varphi)$
- instead of adding $(B_i \leftrightarrow \varphi_i)$, add $(B_i^+ \rightarrow \varphi_i^+) \wedge (B_i^- \rightarrow \varphi_i^-)$
- the clauses defining B_i^+, B_i^- can be satisfied by $\eta^{\mathbf{B}}(B_i^+) = \eta^{\mathbf{B}}(B_i^-) = \bot$ without assigning a truth value to φ_i

Theorem (see extended version of this paper [3])

$$\mu^{\mathbf{A}} \models \varphi \implies \mu^{\mathbf{A}} \models \exists \mathbf{B}.\mathsf{CNF_{PG}}(\mathsf{NNF}(\varphi))$$

i.e., there exists $\eta^{\mathbf{B}}$ s.t. $\mu^{\mathbf{A}} \cup \eta^{\mathbf{B}} \models \mathsf{CNF_{PG}}(\mathsf{NNF}(\varphi))$

- the viceversa holds trivially

[3] G. Masina, G. Spallitta, and R. Sebastiani (2023-06). *On CNF Conversion for SAT Enumeration.* arXiv: 2303.14971

Combining NNF and Plaisted and Greenbaum CNF

$\mathsf{NNF} + \mathsf{CNF}_{\mathsf{PG}}$ allows for assigning "**don't care**" value also to φ_is with **double polarity**:

- $\varphi_i^+ \stackrel{\text{def}}{=} \mathsf{NNF}(\varphi_i)$ and $\varphi_i^- \stackrel{\text{def}}{=} \mathsf{NNF}(\neg\varphi_i)$ occur only positively in $\mathsf{NNF}(\varphi)$
- instead of adding $(B_i \leftrightarrow \varphi_i)$, add $(B_i^+ \to \varphi_i^+) \wedge (B_i^- \to \varphi_i^-)$
- the clauses defining B_i^+, B_i^- can be satisfied by $\eta^{\mathbf{B}}(B_i^+) = \eta^{\mathbf{B}}(B_i^-) = \perp$ without assigning a truth value to φ_i

Theorem (see extended version of this paper [3])

$$\mu^{\mathbf{A}} \models \varphi \implies \mu^{\mathbf{A}} \models \exists \mathbf{B}.\mathsf{CNF}_{\mathsf{PG}}(\mathsf{NNF}(\varphi))$$

i.e., there exists $\eta^{\mathbf{B}}$ s.t. $\mu^{\mathbf{A}} \cup \eta^{\mathbf{B}} \models \mathsf{CNF}_{\mathsf{PG}}(\mathsf{NNF}(\varphi))$

- the viceversa holds trivially

$\eta^{\mathbf{B}}$ is defined as follows:

- if φ_i is made $\top$ by $\mu^{\mathbf{A}}$, then set $\eta^{\mathbf{B}}(B_i^+) = \top$ and $\eta^{\mathbf{B}}(B_i^-) = \perp$
- if φ_i is made $\perp$ by $\mu^{\mathbf{A}}$, then set $\eta^{\mathbf{B}}(B_i^+) = \perp$ and $\eta^{\mathbf{B}}(B_i^-) = \top$
- otherwhise, set $\eta^{\mathbf{B}}(B_i^+) = \eta^{\mathbf{B}}(B_i^-) = \perp$

[3] G. Masina, G. Spallitta, and R. Sebastiani (2023-06). *On CNF Conversion for SAT Enumeration.* arXiv: 2303.14971

Combining NNF and Plaisted and Greenbaum CNF

Remarks

- The result is agnostic w.r.t. the disjoint-AllSAT algorithm adopted.

NN[…] […]ty:

Theorem (see extended version of this paper[3])

$$\mu^{\mathbf{A}} \models \varphi \implies \mu^{\mathbf{A}} \models \exists \mathbf{B}.\mathsf{CNF}_{\mathsf{PG}}(\mathsf{NNF}(\varphi))$$

i.e., there exists $\eta^{\mathbf{B}}$ s.t. $\mu^{\mathbf{A}} \cup \eta^{\mathbf{B}} \models \mathsf{CNF}_{\mathsf{PG}}(\mathsf{NNF}(\varphi))$

- the viceversa holds trivially

$\eta^{\mathbf{B}}$ is defined as follows:

- if φ_i is made $\top$ by $\mu^{\mathbf{A}}$, then set $\eta^{\mathbf{B}}(B_i^+) = \top$ and $\eta^{\mathbf{B}}(B_i^-) = \bot$
- if φ_i is made $\bot$ by $\mu^{\mathbf{A}}$, then set $\eta^{\mathbf{B}}(B_i^+) = \bot$ and $\eta^{\mathbf{B}}(B_i^-) = \top$
- otherwhise, set $\eta^{\mathbf{B}}(B_i^+) = \eta^{\mathbf{B}}(B_i^-) = \bot$

[3] G. Masina, G. Spallitta, and R. Sebastiani (2023-06). *On CNF Conversion for SAT Enumeration.* arXiv: 2303.14971

Combining NNF and Plaisted and Greenbaum CNF

Remarks

- The result is agnostic w.r.t. the disjoint-AllSAT algorithm adopted.
- The result does not guarantee that the enumeration procedure always finds these extended models, but only that these models exist.

NN… …ty:

Theorem (see extended version of this paper[3])

$$\mu^{\mathbf{A}} \models \varphi \implies \mu^{\mathbf{A}} \models \exists \mathbf{B}.\mathsf{CNF}_{\mathsf{PG}}(\mathsf{NNF}(\varphi))$$

i.e., there exists $\eta^{\mathbf{B}}$ s.t. $\mu^{\mathbf{A}} \cup \eta^{\mathbf{B}} \models \mathsf{CNF}_{\mathsf{PG}}(\mathsf{NNF}(\varphi))$

- the viceversa holds trivially

$\eta^{\mathbf{B}}$ is defined as follows:

- if φ_i is made $\top$ by $\mu^{\mathbf{A}}$, then set $\eta^{\mathbf{B}}(B_i^+) = \top$ and $\eta^{\mathbf{B}}(B_i^-) = \bot$
- if φ_i is made $\bot$ by $\mu^{\mathbf{A}}$, then set $\eta^{\mathbf{B}}(B_i^+) = \bot$ and $\eta^{\mathbf{B}}(B_i^-) = \top$
- otherwhise, set $\eta^{\mathbf{B}}(B_i^+) = \eta^{\mathbf{B}}(B_i^-) = \bot$

[3] G. Masina, G. Spallitta, and R. Sebastiani (2023-06). *On CNF Conversion for SAT Enumeration.* arXiv: 2303.14971

Combining NNF and Plaisted and Greenbaum CNF

Remarks

- The result is agnostic w.r.t. the disjoint-AllSAT algorithm adopted.
- The result does not guarantee that the enumeration procedure always finds these extended models, but only that these models exist.
  - Approximated by instructing the solver to split on negative value first

NNF ... ty:

Theorem (see extended version of this paper [3])

$$\mu^{\mathbf{A}} \models \varphi \implies \mu^{\mathbf{A}} \models \exists \mathbf{B}.\mathsf{CNF}_{\mathsf{PG}}(\mathsf{NNF}(\varphi))$$

i.e., there exists $\eta^{\mathbf{B}}$ s.t. $\mu^{\mathbf{A}} \cup \eta^{\mathbf{B}} \models \mathsf{CNF}_{\mathsf{PG}}(\mathsf{NNF}(\varphi))$

- the viceversa holds trivially

$\eta^{\mathbf{B}}$ is defined as follows:

- if φ_i is made $\top$ by $\mu^{\mathbf{A}}$, then set $\eta^{\mathbf{B}}(B_i^+) = \top$ and $\eta^{\mathbf{B}}(B_i^-) = \bot$
- if φ_i is made $\bot$ by $\mu^{\mathbf{A}}$, then set $\eta^{\mathbf{B}}(B_i^+) = \bot$ and $\eta^{\mathbf{B}}(B_i^-) = \top$
- otherwhise, set $\eta^{\mathbf{B}}(B_i^+) = \eta^{\mathbf{B}}(B_i^-) = \bot$

[3] G. Masina, G. Spallitta, and R. Sebastiani (2023-06). *On CNF Conversion for SAT Enumeration.* arXiv: 2303.14971

Combining NNF and Plaisted and Greenbaum CNF

NNF ... ty:

Remarks

- The result is agnostic w.r.t. the disjoint-AllSAT algorithm adopted.
- The result does not guarantee that the enumeration procedure always finds these extended models, but only that these models exist.
  - Approximated by instructing the solver to split on negative value first
  - Ad-hoc heuristics should be investigated.

Theorem (see extended version of this paper[3])

$$\mu^{\mathbf{A}} \models \varphi \implies \mu^{\mathbf{A}} \models \exists \mathbf{B}.\mathsf{CNF}_{\mathsf{PG}}(\mathsf{NNF}(\varphi))$$

i.e., there exists $\eta^{\mathbf{B}}$ s.t. $\mu^{\mathbf{A}} \cup \eta^{\mathbf{B}} \models \mathsf{CNF}_{\mathsf{PG}}(\mathsf{NNF}(\varphi))$

- the viceversa holds trivially

$\eta^{\mathbf{B}}$ is defined as follows:

- if φ_i is made $\top$ by $\mu^{\mathbf{A}}$, then set $\eta^{\mathbf{B}}(B_i^+) = \top$ and $\eta^{\mathbf{B}}(B_i^-) = \bot$
- if φ_i is made $\bot$ by $\mu^{\mathbf{A}}$, then set $\eta^{\mathbf{B}}(B_i^+) = \bot$ and $\eta^{\mathbf{B}}(B_i^-) = \top$
- otherwhise, set $\eta^{\mathbf{B}}(B_i^+) = \eta^{\mathbf{B}}(B_i^-) = \bot$

[3] G. Masina, G. Spallitta, and R. Sebastiani (2023-06). *On CNF Conversion for SAT Enumeration.* arXiv: 2303.14971

Outline

Experimental settings

Settings

- We implemented each CNF-ization algorithm
- We use MathSAT to perform projected enumeration
  - forced to split on negative value first in decision branches

Problem sets

- Synthetic benchmarks: random Boolean formulas generated by nesting Boolean operators up to a fixed depth
- Real-world-inspired benchmarks:
  - Problems encoding properties ISCAS'85 circuits as in [4]
  - Weighted Model Integration instances as in [5]

[4] A. T. Tibebu and G. Fey (2018). "Augmenting All Solution SAT Solving for Circuits with Structural Information". In: DDECS

[5] G. Spallitta, G. Masina, P. Morettin, A. Passerini, and R. Sebastiani (2022). "SMT-based Weighted Model Integration with Structure Awareness". In: UAI

Experimental results - Synthetic benchmark

of models:

Execution time:

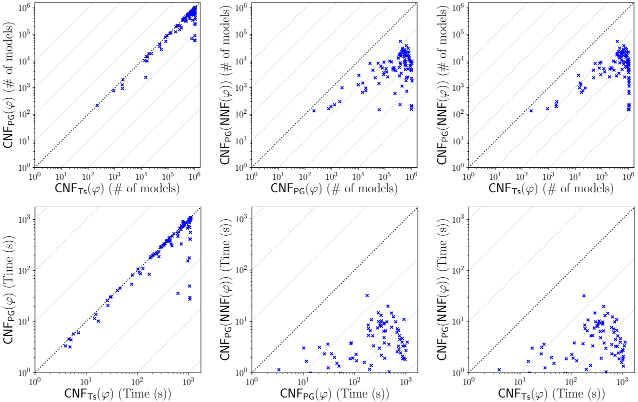


Figure 1: 100 instances, each with 20 Boolean atoms and depth 8.
All the axes are on a logarithmic scale. In these problems, there were no timeouts.

Experimental results - Circuits from ISCAS'85

of models:

Execution time:

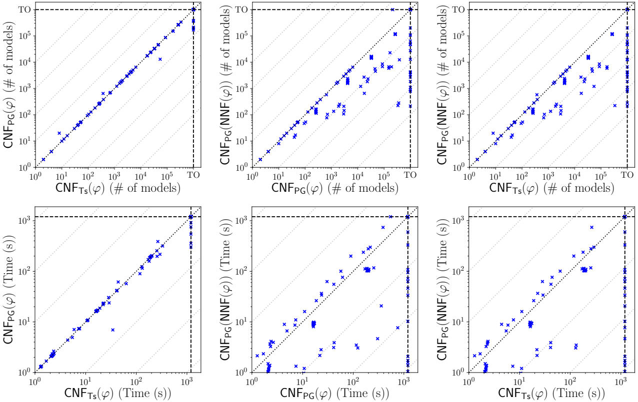


Figure 2: 250 instances, circuits have from 32 to 202 inputs.
All the axes are on a logarithmic scale. In these problems, the $\mathsf{CNF_{Ts}}$, $\mathsf{CNF_{PG}}$ and $\mathsf{NNF} + \mathsf{CNF_{PG}}$ reported 49, 44 and 27 timeouts, respectively, represented by the points on the dashed lines.

Experimental results - Weighted Model Integration

\# of models:

Execution time:

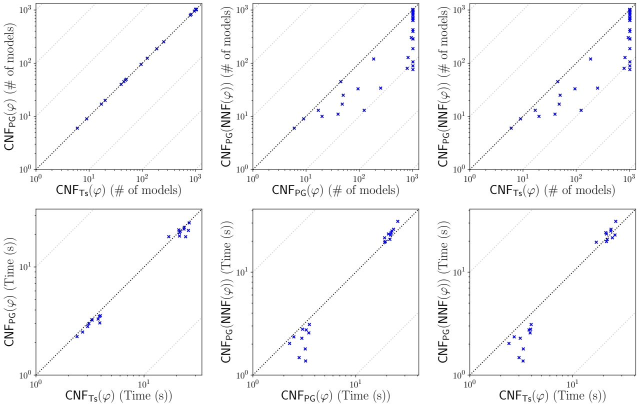


Figure 3: 40 instances, 10 Boolean variables, depths 3, 5, 7, 9.
All the axes are on a logarithmic scale. In these problems, there were no timeouts.

Outline

Conclusions

We have presented a theoretical and empirical analysis of the impact of different CNF-ization approaches on SAT enumeration:

- $\mathsf{CNF}_{\mathsf{Ts}}$ and $\mathsf{CNF}_{\mathsf{PG}}$ prevent the solver from producing *minimal* partial assignments
- $\mathsf{NNF}+\mathsf{CNF}_{\mathsf{PG}}$ can fully overcome the problem and drastically reduce the number of partial assignments and the execution time

Ongoing Work

- Study the impact of CNF-ization on:
  - non-disjoint SAT enumeration
  - disjoint SMT enumeration

    $\Longrightarrow$ applications in WMI and #SMT
- Elaborate novel disjoint AllSAT/AllSMT procedures to best exploit this encoding

Thanks for your attention!

Let $\mathbf{A} \stackrel{\text{def}}{=} \{A_1, \ldots, A_4\}, \quad \mathbf{B} \stackrel{\text{def}}{=} \{B_1, \ldots, B_3\}$

$$\begin{aligned}\varphi(\mathbf{A} \cup \mathbf{B}) \stackrel{\text{def}}{=} & (\neg B_1 \vee A_1) \wedge (\neg B_1 \vee A_1) \wedge (\neg B_2 \vee A_3) \wedge (\neg B_2 \vee A_4) \wedge \\ & (\neg B_3 \vee A_2 \vee B_2) \wedge (B_1 \vee B_2)\end{aligned}$$

[6] S. K. Lahiri, R. Nieuwenhuis, and A. Oliveras (2006). "SMT Techniques for Fast Predicate Abstraction". In: CAV

[7] A. Cimatti, A. Griggio, B. J. Schaafsma, and R. Sebastiani (2013). "The MathSAT5 SMT Solver". In: TACAS

Let $\mathbf{A} \stackrel{\text{def}}{=} \{A_1, \ldots, A_4\}, \quad \mathbf{B} \stackrel{\text{def}}{=} \{B_1, \ldots, B_3\}$

$$\varphi(\mathbf{A} \cup \mathbf{B}) \stackrel{\text{def}}{=} (\neg B_1 \vee A_1) \wedge (\neg B_1 \vee A_1) \wedge (\neg B_2 \vee A_3) \wedge (\neg B_2 \vee A_4) \wedge$$
$$(\neg B_3 \vee A_2 \vee B_2) \wedge (B_1 \vee B_2)$$

A simple enumeration strategy [6] (e.g. in MathSAT [7]):

[6] S. K. Lahiri, R. Nieuwenhuis, and A. Oliveras (2006). "SMT Techniques for Fast Predicate Abstraction". In: CAV

[7] A. Cimatti, A. Griggio, B. J. Schaafsma, and R. Sebastiani (2013). "The MathSAT5 SMT Solver". In: TACAS

Let $\mathbf{A} \stackrel{\text{def}}{=} \{A_1, \ldots, A_4\}, \quad \mathbf{B} \stackrel{\text{def}}{=} \{B_1, \ldots, B_3\}$

$$\varphi(\mathbf{A} \cup \mathbf{B}) \stackrel{\text{def}}{=} (\neg B_1 \vee A_1) \wedge (\neg B_1 \vee A_1) \wedge (\neg B_2 \vee A_3) \wedge (\neg B_2 \vee A_4) \wedge$$
$$(\neg B_3 \vee A_2 \vee B_2) \wedge (B_1 \vee B_2)$$

A simple enumeration strategy [6] (e.g. in MathSAT [7]):

1. find a total truth assignment $\eta \stackrel{\text{def}}{=} \eta^{\mathbf{A}} \cup \eta^{\mathbf{B}}$ s.t. $\eta \models \varphi$, e.g.:

$$\eta \stackrel{\text{def}}{=} \{\underbrace{A_1, A_2, A_3, A_4}_{\eta^{\mathbf{A}}}, \underbrace{B_1, \neg B_3, \neg B_2}_{\eta^{\mathbf{B}}}\}$$

[6] S. K. Lahiri, R. Nieuwenhuis, and A. Oliveras (2006). "SMT Techniques for Fast Predicate Abstraction". In: CAV

[7] A. Cimatti, A. Griggio, B. J. Schaafsma, and R. Sebastiani (2013). "The MathSAT5 SMT Solver". In: TACAS

Let $\mathbf{A} \stackrel{\text{def}}{=} \{A_1, \ldots, A_4\}, \quad \mathbf{B} \stackrel{\text{def}}{=} \{B_1, \ldots, B_3\}$

$$\varphi(\mathbf{A} \cup \mathbf{B}) \stackrel{\text{def}}{=} (\neg B_1 \vee A_1) \wedge (\neg B_1 \vee A_1) \wedge (\neg B_2 \vee A_3) \wedge (\neg B_2 \vee A_4) \wedge$$
$$(\neg B_3 \vee A_2 \vee B_2) \wedge (B_1 \vee B_2)$$

A simple enumeration strategy [6] (e.g. in MathSAT [7]):

1. find a total truth assignment $\eta \stackrel{\text{def}}{=} \eta^{\mathbf{A}} \cup \eta^{\mathbf{B}}$ s.t. $\eta \models \varphi$, e.g.:

$$\eta \stackrel{\text{def}}{=} \{\underbrace{A_1, A_2, A_3, A_4}_{\eta^{\mathbf{A}}}, \underbrace{B_1, \neg B_3, \neg B_2}_{\eta^{\mathbf{B}}}\}$$

2. find a minimal $\mu^{\mathbf{A}} \subseteq \eta^{\mathbf{A}}$ s.t. $\mu^{\mathbf{A}} \cup \eta^{\mathbf{B}} \models \varphi$, e.g.:

$$\mu \stackrel{\text{def}}{=} \{\underbrace{A_1, A_2}_{\mu^{\mathbf{A}}}, \underbrace{B_1, \neg B_3, \neg B_2}_{\eta^{\mathbf{B}}}\}$$

[6] S. K. Lahiri, R. Nieuwenhuis, and A. Oliveras (2006). "SMT Techniques for Fast Predicate Abstraction". In: CAV

[7] A. Cimatti, A. Griggio, B. J. Schaafsma, and R. Sebastiani (2013). "The MathSAT5 SMT Solver". In: TACAS

Let $\mathbf{A} \stackrel{\text{def}}{=} \{A_1, \ldots, A_4\}, \quad \mathbf{B} \stackrel{\text{def}}{=} \{B_1, \ldots, B_3\}$

$$\varphi(\mathbf{A} \cup \mathbf{B}) \stackrel{\text{def}}{=} (\neg B_1 \vee A_1) \wedge (\neg B_1 \vee A_1) \wedge (\neg B_2 \vee A_3) \wedge (\neg B_2 \vee A_4) \wedge$$
$$(\neg B_3 \vee A_2 \vee B_2) \wedge (B_1 \vee B_2) \wedge (\neg A_1 \vee \neg A_2)$$

A simple enumeration strategy [6] (e.g. in MathSAT [7]):

1. find a total truth assignment $\eta \stackrel{\text{def}}{=} \eta^{\mathbf{A}} \cup \eta^{\mathbf{B}}$ s.t. $\eta \models \varphi$, e.g.:

$$\eta \stackrel{\text{def}}{=} \{\underbrace{A_1, A_2, A_3, A_4}_{\eta^{\mathbf{A}}}, \underbrace{B_1, \neg B_3, \neg B_2}_{\eta^{\mathbf{B}}}\}$$

2. find a minimal $\mu^{\mathbf{A}} \subseteq \eta^{\mathbf{A}}$ s.t. $\mu^{\mathbf{A}} \cup \eta^{\mathbf{B}} \models \varphi$, e.g.:

$$\mu \stackrel{\text{def}}{=} \{\underbrace{A_1, A_2}_{\mu^{\mathbf{A}}}, \underbrace{B_1, \neg B_3, \neg B_2}_{\eta^{\mathbf{B}}}\}$$

3. Learn the blocking clause $\neg \mu^{\mathbf{A}}$ to ensure the disjointness and continue the search

[6] S. K. Lahiri, R. Nieuwenhuis, and A. Oliveras (2006). "SMT Techniques for Fast Predicate Abstraction". In: CAV

[7] A. Cimatti, A. Griggio, B. J. Schaafsma, and R. Sebastiani (2013). "The MathSAT5 SMT Solver". In: TACAS

Minimization procedure

Algorithm 1 MINIMIZE-ASSIGNMENT$(\psi_i, \eta_i, \mathbf{A})$ // $\psi_i \overset{\text{def}}{=} \psi \wedge \bigwedge_{j=1}^{i-1} \neg\mu_j^{\mathbf{A}}$, $\eta_i = \eta_i^{\mathbf{A}} \cup \eta_i^{\mathbf{B}}$

$\mu_i^{\mathbf{A}} \leftarrow \eta_i^{\mathbf{A}}$
for $\ell \in \mu_i^{\mathbf{A}}$ **do**
  if $\psi_i|_{\mu_i^{\mathbf{A}} \setminus \{\ell\} \,\cup\, \eta_i^{\mathbf{B}}} = \top$ **then**
    $\mu_i^{\mathbf{A}} \leftarrow \mu_i^{\mathbf{A}} \setminus \{\ell\}$
return $\mu_i^{\mathbf{A}}$

Iteratively drop literals one-by-one from $\eta_i^{\mathbf{A}}$, checking if it still satisfies the formula.

Each minimization step is $O(\#\mathit{clauses} \cdot \#\mathit{vars})$

NNF DAG

Property

If $\mathsf{NNF}(\varphi)$ is represented as a DAG, then its size is linear w.r.t. the size of φ.

NNF DAG

Property

If $\mathsf{NNF}(\varphi)$ is represented as a DAG, then its size is linear w.r.t. the size of φ.

Intuition:

- for each subformula φ_i, the DAG has at most 2 nodes for $\mathsf{NNF}(\varphi_i)$ and $\mathsf{NNF}(\neg\varphi_i)$ (positive and negative occurrences of φ_i resp.)
- nodes shared by up to exponentially-many branches generated by expanding nested $\leftrightarrow$

NNF DAG

Property

If $\mathsf{NNF}(\varphi)$ is represented as a DAG, then its size is linear w.r.t. the size of φ.

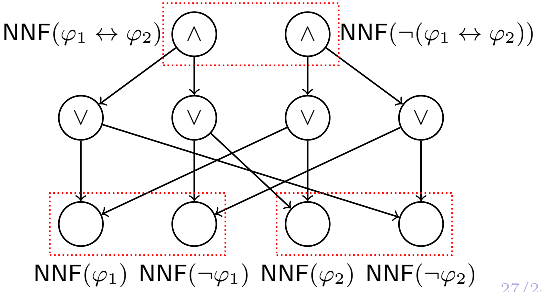
Intuition:

- for each subformula φ_i, the DAG has at most 2 nodes for $\mathsf{NNF}(\varphi_i)$ and $\mathsf{NNF}(\neg\varphi_i)$ (positive and negative occurrences of φ_i resp.)
- nodes shared by up to exponentially-many branches generated by expanding nested $\leftrightarrow$

Proof sketch

- Prove that the 2-root DAG for the pair $\langle\mathsf{NNF}(\varphi), \mathsf{NNF}(\neg\varphi)\rangle$ grows linearly w.r.t. $|\varphi|$
- E.g., if $\varphi \stackrel{\text{def}}{=} \varphi_1 \leftrightarrow \varphi_2$, then

$$
\begin{aligned}
|\langle\mathsf{NNF}(\varphi), \mathsf{NNF}(\neg\varphi)\rangle| = \quad & 18\ + \\
& |\langle\mathsf{NNF}(\varphi_1), \mathsf{NNF}(\neg\varphi_1)\rangle|\ + \\
& |\langle\mathsf{NNF}(\varphi_2), \mathsf{NNF}(\neg\varphi_2)\rangle|
\end{aligned}
$$



Size of $\mathsf{CNF}_{\mathsf{PG}}(\mathsf{NNF}(\varphi))$

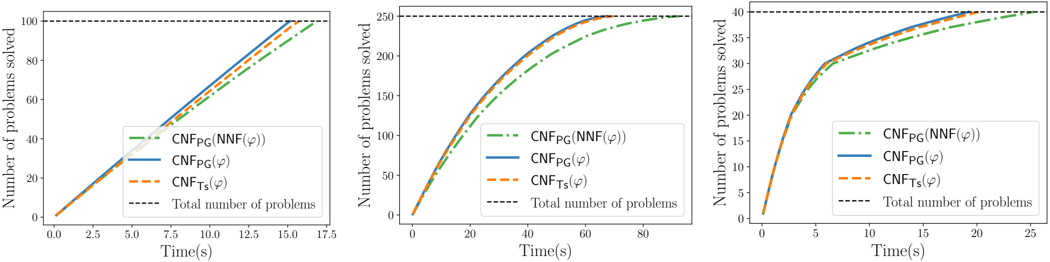
- The NNF DAG contains at most 2 nodes for each subformula φ_i of φ.
- For each node of the NNF DAG —representing the sub-formula φ_i— $\mathsf{CNF}_{\mathsf{PG}}$ introduces:
  - One variable B_i
  - One or two clauses representing $(B_i \rightarrow \varphi_i)$

Hence, the size of $\mathsf{CNF}_{\mathsf{PG}}(\mathsf{NNF}(\varphi))$ is linear w.r.t. the size of φ.

Results for SAT solving

Remark

The pre-conversion into NNF is typically not used in SAT *solving*, because the unnecessary duplication of the labels B_i^+ and B_i^- causes extra overhead and no benefit for the solver.



(a) Synthetic benchmarks (b) Circuit benchmarks (c) WMI benchmarks

Figure 4: The y-axis reports the instances for which the solver finished within the cumulative time on the x-axis.

Disjoint AllSAT/AllSMT applications

#SMT($\mathcal{LRA}$)

Given an SMT($\mathcal{LRA}$) formula $\varphi(\mathbf{A}, \mathbf{x})$ [8]

$$\#\text{SMT}(\mathcal{LRA})(\varphi|\mathbf{A},\mathbf{x}) \stackrel{\text{def}}{=} \sum_{\mu^{\mathbf{A}} \cup \mu^{\mathcal{LRA}} \in \mathcal{TA}(\varphi)} \left(2^{|\mathbf{A}|-|\mu^{\mathbf{A}}|} \cdot vol(\mu^{\mathcal{LRA}})\right)$$

Weighted Model Integration

Given an SMT($\mathcal{LRA}$) formula $\varphi(\mathbf{A}, \mathbf{x})$ and a weight function $w(\mathbf{A}, \mathbf{x})$ [9] [10] [11]

$$\mathsf{WMI}(\varphi, w|\mathbf{A},\mathbf{x}) \stackrel{\text{def}}{=} \sum_{\mu^{\mathbf{A}} \cup \mu^{\mathcal{LRA}} \in \mathcal{TA}(\varphi)} \left(2^{|\mathbf{A}|-|\mu^{\mathbf{A}}|} \cdot \int_{\mu^{\mathcal{LRA}}} w(\mathbf{x}|\mathbf{A}) d\mathbf{x}\right)$$

[8] D. Chistikov et al. (2015). “Approximate Counting in SMT and Value Estimation for Probabilistic Programs”. In: TACAS

[9] P. Morettin et al. (2017). “Efficient Weighted Model Integration via SMT-Based Predicate Abstraction”. In: IJCAI

[10] P. Morettin et al. (2019). “Advanced SMT techniques for Weighted Model Integration”. In: *AIJ*

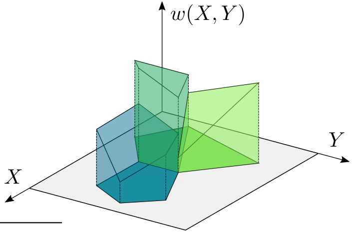
[11] G. Spallitta et al. (2022). “SMT-based Weighted Model Integration with Structure Awareness”. In: UAI

WMI

Weighted Model Integration

Given an SMT($\mathcal{LRA}$) formula $\varphi(\mathbf{A}, \mathbf{x})$ and a weight function $w(\mathbf{A}, \mathbf{x})$ [12] [13] [14]

$$\mathsf{WMI}(\varphi, w|\mathbf{A}, \mathbf{x}) \stackrel{\text{def}}{=} \sum_{\mu^{\mathbf{A}} \cup \mu^{\mathcal{LRA}} \in \mathcal{TA}(\varphi)} \left(2^{|\mathbf{A}|-|\mu^{\mathbf{A}}|} \cdot \int_{\mu^{\mathcal{LRA}}} w(\mathbf{x}|\mathbf{A}) d\mathbf{x} \right)$$



[12] P. Morettin, A. Passerini, and R. Sebastiani (2017). “Efficient Weighted Model Integration via SMT-Based Predicate Abstraction”. In: IJCAI

[13] P. Morettin, A. Passerini, and R. Sebastiani (2019). “Advanced SMT techniques for Weighted Model Integration”. In: *AIJ*

[14] G. Spallitta, G. Masina, P. Morettin, A. Passerini, and R. Sebastiani (2022). “SMT-based Weighted Model Integration with Structure Awareness”. In: UAI

Bibliography I

- Chistikov, D., R. Dimitrova, and R. Majumdar (2015). "Approximate Counting in SMT and Value Estimation for Probabilistic Programs". In: TACAS.
- Cimatti, A., A. Griggio, B. J. Schaafsma, and R. Sebastiani (2013). "The MathSAT5 SMT Solver". In: TACAS.
- Lahiri, S. K., R. Nieuwenhuis, and A. Oliveras (2006). "SMT Techniques for Fast Predicate Abstraction". In: CAV.
- Masina, G., G. Spallitta, and R. Sebastiani (2023-06). *On CNF Conversion for SAT Enumeration*. arXiv: 2303.14971.
- Morettin, P., A. Passerini, and R. Sebastiani (2017). "Efficient Weighted Model Integration via SMT-Based Predicate Abstraction". In: IJCAI.
- Morettin, P., A. Passerini, and R. Sebastiani (2019). "Advanced SMT techniques for Weighted Model Integration". In: *AIJ*.
- Möhle, S., R. Sebastiani, and A. Biere (2020). "Four Flavors of Entailment". In: SAT.
- Sebastiani, R. (2020). *Are You Satisfied by This Partial Assignment?* arXiv: 2003.04225.

Bibliography II

Spallitta, G., G. Masina, P. Morettin, A. Passerini, and R. Sebastiani (2022). "SMT-based Weighted Model Integration with Structure Awareness". In: UAI.

Tibebu, A. T. and G. Fey (2018). "Augmenting All Solution SAT Solving for Circuits with Structural Information". In: DDECS.